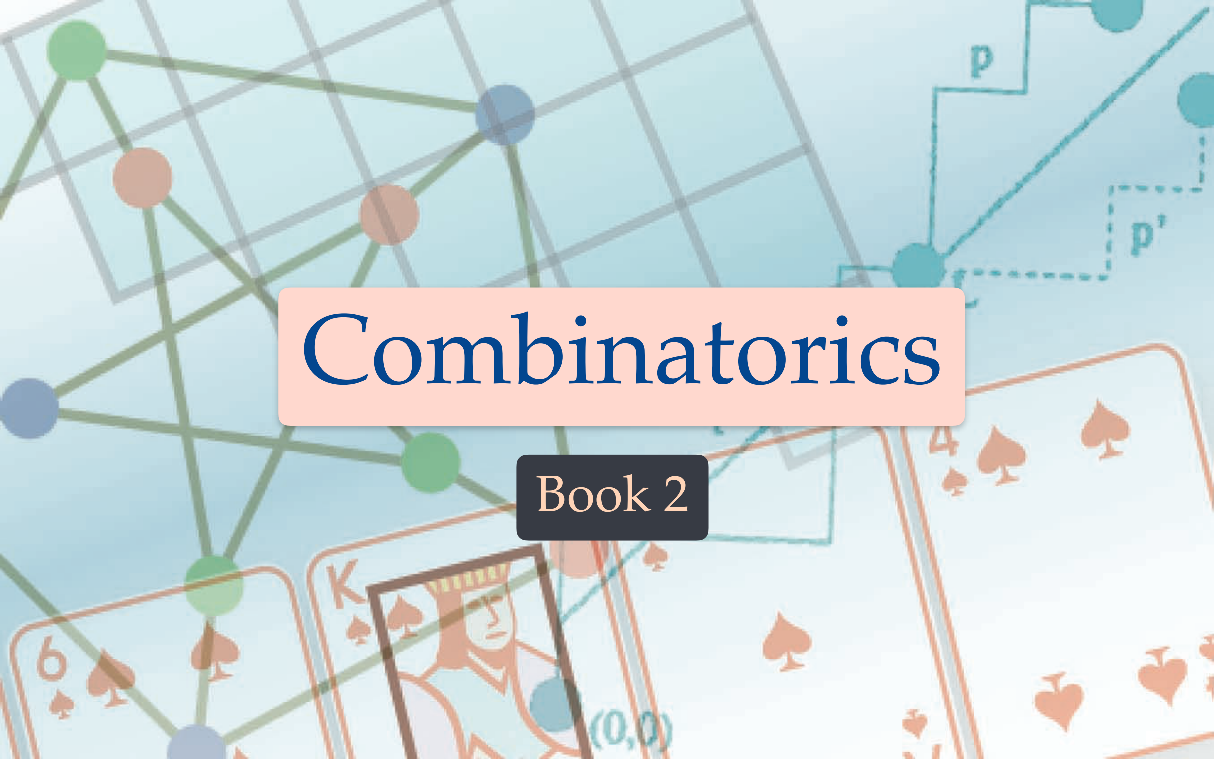


Combinatorics

Book 2

The graph $\Gamma \setminus v$ (formed by removing v and its edges from Γ) has one fewer vertex, so it can be properly colored using at most 6 colors. And since v has at most 5 neighbors in $\Gamma \setminus v$, there is a color left over which can be used to color vertex v. This gives a proper coloring of Γ using at most 6 colors (a contradiction...)

We will improve this to show that actually 5 colors suffice to properly color every planar graph.

$\psi \; \phi \; \chi \; \pi$

A b c ... x X χ

Greek "chi"

Given a graph Γ, the chromatic number of Γ, denoted $\chi(\Gamma)$, is the smallest number of colors we can use to properly color the vertices of Γ. A proper coloring of the vertices of Γ is a coloring of the vertices such that no edge has both endpoints of the same color.

The theorem of Appel and Haken (1976) is that every planar graph Γ has $\chi(\Gamma) \leq 4$.

Note that $\chi(K_n) = n$. Here K_n is the complete graph of order n.

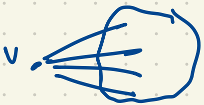
A graph Γ has $\chi(\Gamma) = 1$ iff it has vertices but no edges.

A graph Γ has $\chi(\Gamma) \leq 2$ iff Γ is bipartite iff Γ has no circuits of odd length.

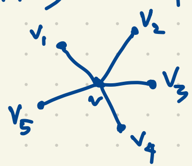
Computing $\chi(\Gamma)$ is hard in general.

Theorem If Γ is a finite planar graph then $\chi(\Gamma) \leq 5$. Proof due to Heawood.

Proof If the theorem fails then there is a smallest counterexample Γ with n vertices (so Γ is planar and every planar graph of order $n-1$ has chromatic number ≤ 5 while $\chi(\Gamma) \geq 6$). We seek a contradiction. Γ has a vertex v of degree ≤ 5. In fact $\deg v = 5$. (If $\deg v \leq 4$ then $\chi(\Gamma) \leq 5$, a contradiction.) Let Γ' be the graph obtained from Γ by deleting v and its five edges,

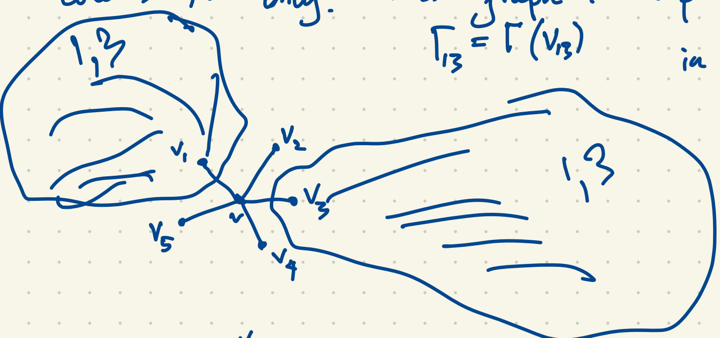


$n-1$ vertices can be properly colored using at most five colors.

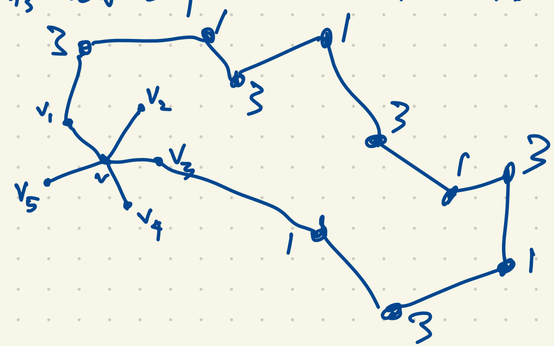


so $\chi(\Gamma') \leq 5$. Say v_i has color i $(i=1,2,\cdots,5)$.

Consider the vertices $V_{13} \subset \{\text{vertices of } \Gamma\}$ having colors 1, 3 only. This graph is bipartite. I can assume v_1 is joined to v_3 in Γ_{13} (otherwise
$\Gamma_{13} = \Gamma(V_{13})$
in part of Γ_{13}, reverse colors 1, 3 so that v_3 gets color 1. Then we are free to color v using color 3 since its neighbors are color 1, 2, 1, 4, 5).

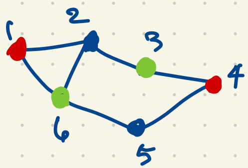


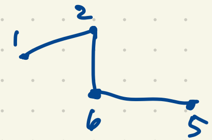
Otherwise Γ_{13} has a path from v_1 to v_3.



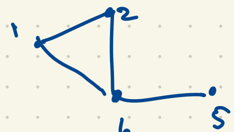
Similarly there is a path from v_2 to v_4 using only vertices of colors 2 and 4. Contradiction! $\square$

Given a graph Γ, a <u>subgraph</u> of Γ is formed by taking a subset of the edges of Γ together with all their vertices. An <u>induced subgraph</u> of Γ is formed by taking a subset of the vertices of Γ together with all their edges in Γ

$\Gamma =$ 

eg. 

is a subgraph of Γ. (not an induced subgraph of Γ)

eg. 

is an induced subgraph of Γ.

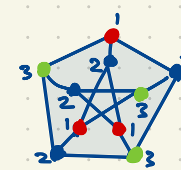
An induced subgraph of Γ is a subgraph of Γ, but not conversely.

A <u>k-clique</u> in Γ is a complete subgraph of Γ, i.e. a subset of the vertices, any two of which are joined.

In Γ above, $\{1,2,6\}$ is a clique (in fact a 3-clique). The <u>clique number</u> of Γ, denoted $\omega(\Gamma)$, is the size of the largest clique in Γ. It is hard to compute $\omega(\Gamma)$.

w vs. ω
Roman Greek

<u>Theorem</u> For every graph Γ, $\chi(\Gamma) \geq \omega(K)$.



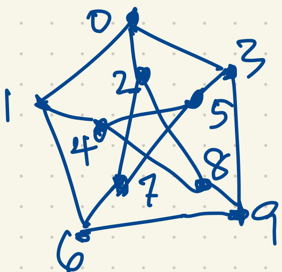
$\chi(P) = 3$ for the Petersen graph P.

Warning: this not equality! For the Petersen graph P, $\omega(P) = 2$.

Proof: The vertices in a clique of size $\omega(\Gamma)$ require $\omega(\Gamma)$ different colors.

Dual to the clique number $\omega(\Gamma)$ we have the coclique number $\alpha(\Gamma)$ which is the maximum number of vertices in Γ, no two of which are joined. (This is $\alpha(\Gamma) = \omega(\bar{\Gamma})$ where $\bar{\Gamma}$ is the complementary graph). Cocliques are also called independent sets of vertices.

Eg. $\alpha(P) = 4$. $|\chi(P)| \geq \frac{10}{4} = 2.5 \Rightarrow \chi(P) \geq 3$.



$\{1,2,3\}$ is a coclique which is not contained in any larger coclique; it is a <u>maximal</u> coclique.

A <u>maximum</u> coclique (i.e. a coclique of maximum size) is $\{1,3,7,8\}$.

This is maximum size because P has vertex set $\{0,3,9,6,1\} \cup \{2,7,5,4,8\}$ as a union of two 5-cycles (circuits of length 5). Any set of size at least 5 vertices has either 3 on the inner 5-cycle $\{2,7,5,4,8\}$ or 3 vertices on the outer 5-cycle $\{0,3,9,6,1\}$. In either case there is an edge in that 5-cycle joining two vertices we have chosen.

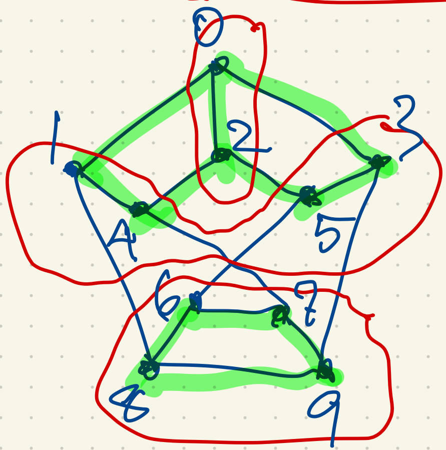
<u>Theorem</u> $\chi(\Gamma) \geq \dfrac{|V|}{\alpha(\Gamma)}$ where $|V|$ = the number of vertices = the order of Γ.

<u>Proof</u> Let $k = \chi(\Gamma)$. Properly color the vertices $1, 2, \cdots, k$ and let V_i be the subset of vertices colored i, for $i = 1, 2, \cdots, k$. This gives a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ ($V_i \cup V_j$ = union of V_i and V_j; $V_i \sqcup V_j$ = disjoint union of V_i and V_j). Each V_i is a coclique so $|V_i| \leq \alpha(\Gamma)$ so $|V| = |V_1| + |V_2| + \cdots + |V_k| \leq \underbrace{\alpha(\Gamma) + \alpha(\Gamma) + \cdots + \alpha(\Gamma)}_{k} = k\,\alpha(\Gamma)$ so $\chi(\Gamma) = k \geq \dfrac{|V|}{\alpha(\Gamma)}$. $\square$

March	M	W	F
		1	3
	6	$\boxed{8}$	10
	13	15	17

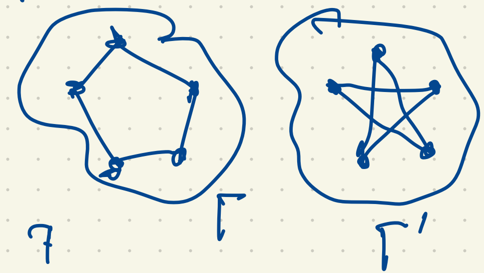
Spring Break

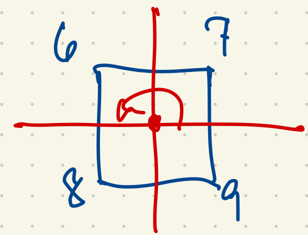
Test 1: Wed Mar 8



$= \Gamma$

You can use nauty to test isomorphism between two graphs.





Using nauty software

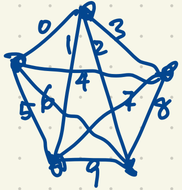
$G = \operatorname{Aut}\Gamma$

$G = \langle (1,3)(4,5)(6,7)(8,9),\ (0,2)(1,4)(3,5)(6,9)(7,8) \rangle$

$|G| = 4$

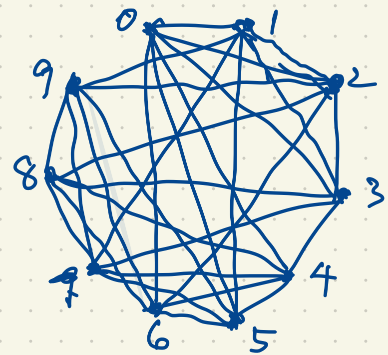
G has 3 orbits on the vertices: $\{0,2\}$, $\{1,3,4,5\}$, $\{6,7,8,9\}$

$K_5 =$

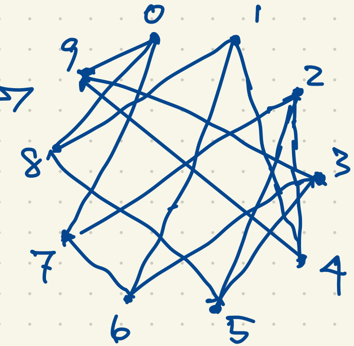


$\longrightarrow$

K_5'

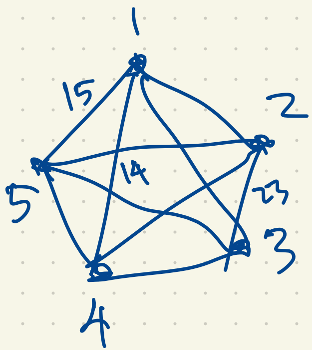


$\xrightarrow{\overline{K_5'}}$

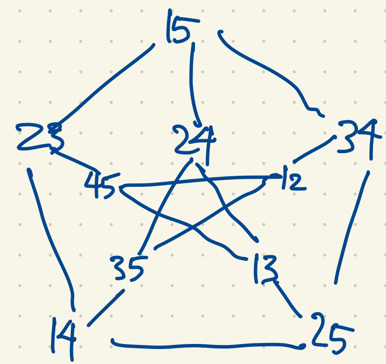


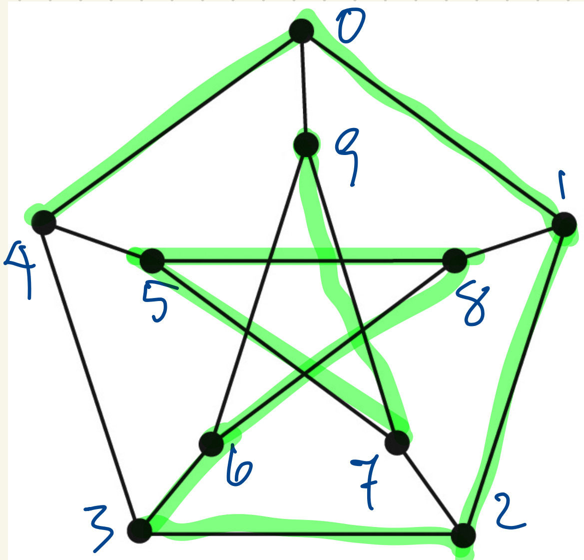
$|\mathrm{Aut}\ \overline{K_5'}| = 120$

$K_5 =$

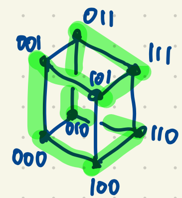


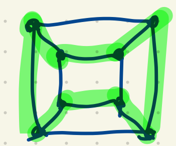
$\overline{K_5'} =$





The Petersen graph P has a Hamilton path $(0,1,2,3,6,8,5,7,9)$ (a path touching each vertex exactly once) but no Hamilton circuit (ending at the same vertex where it started).

The Hamming cube H_3 = 

= 

does have a Hamilton circuit.

000
100
110
010
011
111
101
001
000

"Gray code"

Every Hamming graph H_n $(n \geqslant 2)$ has a Hamilton circuit.

A graph having a Hamilton circuit is called <u>Hamiltonian</u>.

Looking for Hamilton paths or circuits is known to be difficult in general.
Testing whether a giving graph Γ is Hamiltonian is NP-complete.