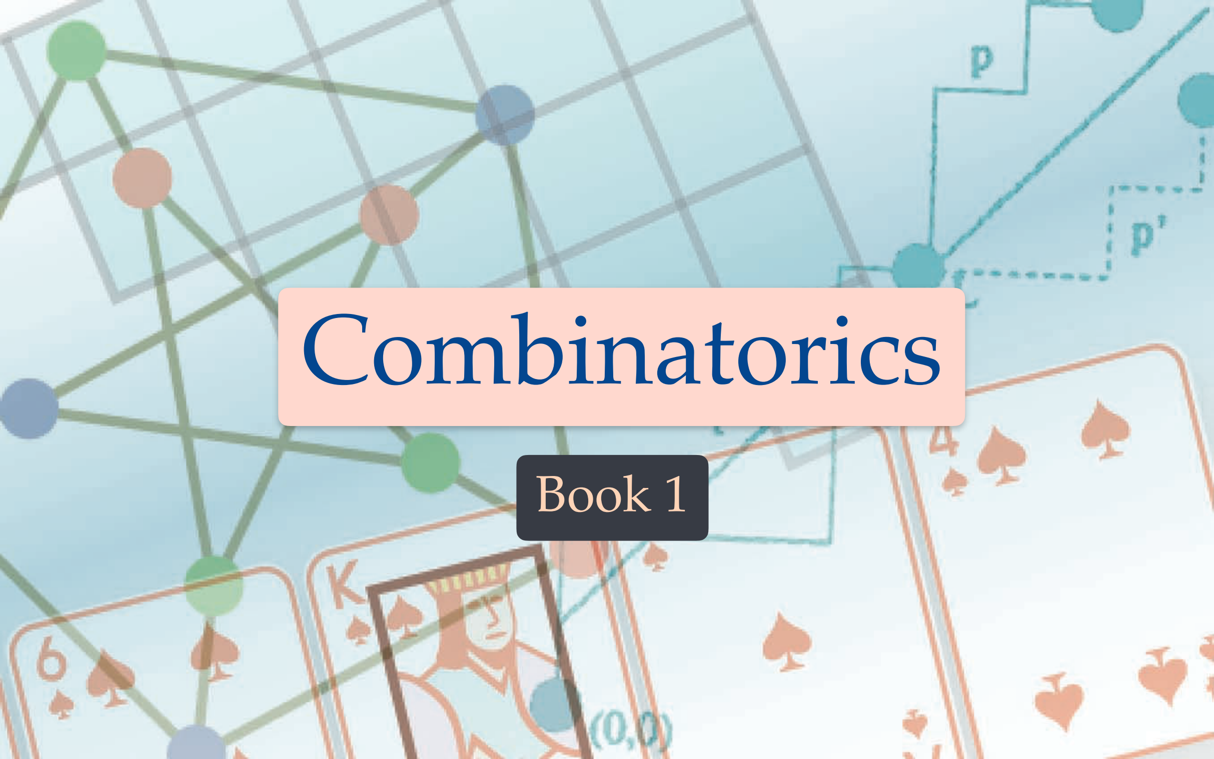
Combinatorics

Book 1



#vertices	Connected graphs	All graphs
1	1	1
2	1	2
3	2	4
4	6	11
5	21	34
6	112	156
7	853	1044
8	11117	12346
9	261080	274668
10	11716571	12005168
11	1006700565	1018997864
12	164059830476	165091172592
13	50335907869219	50502031367952
14	29003487462848061	29054155657235488
15	31397381142761241960	31426485969804308768
16	63969560113225176176277	64001015704527557894928

Eric

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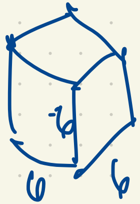
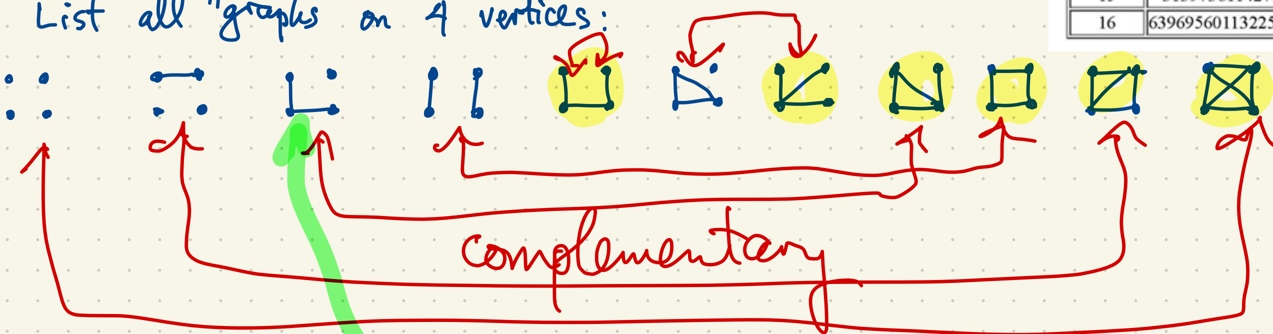


$6^3 = 216$

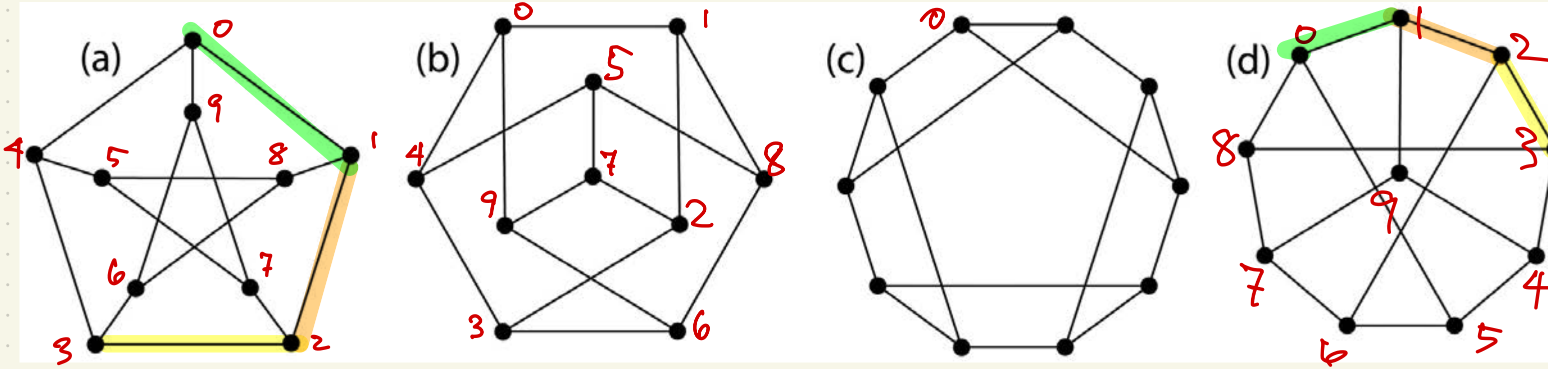
Ordinary / Simple Graph on n vertices/nodes

Eg. List all 11 graphs on 4 vertices:



A graph of order n is a pair $G = (V, E)$ where V is a set of n vertices and E is a subset of pairs $\{v, w\}$ where $v \neq w$, $v, w \in V$. Eg. the graph with vertices 1, 2, 3, 4 and edges $\{1,3\}$, $\{2,3\}$ can be illustrated

(the two graphs are isomorphic).



Of these four graphs, which one is not isomorphic to the others?

Graphs (a), (b) are isomorphic. Graph (c) is not isomorphic to (a) or (b) because graph (a) has diameter 2: any two vertices are at distance at most 2 apart. However, graph (c) has diameter 3.

An automorphism (symmetry) of a graph is an isomorphism from the graph to itself.

An isomorphism from graph (a) to graph (d) is the map with table of values

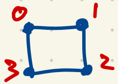
vertex in (a)	vertex in (d)
0	0
1	1
2	2
3	3
4	8
5	7
6	4
7	6
8	9
9	5

This is a very special graph having the special property that for every path of length 3 (vertices v_0, v_1, v_2, v_3 with $v_0 \sim v_1 \sim v_2 \sim v_3$, $v_0 \neq v_2$, $v_0 \neq v_3$, $v_1 \neq v_3$) in (a) and every path $w_0 \sim w_1 \sim w_2 \sim w_3$ in (d) ($w_0 \neq w_2$, $w_0 \neq w_3$, $w_1 \neq w_3$) there is a unique isomorphism (a) $\longrightarrow$ (b) mapping $v_i \mapsto w_i$.

This is a Petersen graph. How many isomorphisms are there from (a) to (d)?

$10 \times 3 \times 2 \times 2 = 120.$

In particular, a Petersen graph has 120 automorphisms.

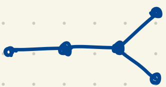
The graph  (a 4-cycle) has 8 automorphisms

$0 \mapsto 1$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 2$	$1 \mapsto 3$	$1 \mapsto 1$
$2 \mapsto 3$	$2 \mapsto 2$	$2 \mapsto 2$
$3 \mapsto 0$	$3 \mapsto 1$	$3 \mapsto 3$
		identity

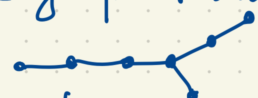
Not an automorphism:

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto 3$
$3 \mapsto 2$

The edge $0 \sim 3$ is mapped to a non-edge $0 \not\sim 2$

The graph  has exactly 2 automorphisms

A graph with only one automorphism?  (the graph of order 1, i.e. having only one vertex).

A less trivial example with more than one vertex:  Γ

Every graph as a degree sequence. The degree of a vertex is the number of its neighbors.

The graph Γ (above) has degree sequence $(1,1,1,2,2,2,3)$.

If two graphs are isomorphic, they must have the same degree sequence.

An isomorphism from Γ to Γ' must map each vertex to a vertex of the same degree.

If two graphs have the same degree sequence, must they be isomorphic?