Combinatorics

Book 2

The graph $\Gamma \setminus v$ (formed by removing v and its edges from Γ) has one fewer vertex, so it can be properly colored using at most 6 colors. And since v has at most 5 neighbors in $\Gamma \setminus v$, there is a color left over which can be used to color vertex v. This gives a proper coloring of Γ using at most 6 colors (a contradiction...)

We will improve this to show that actually 5 colors suffice to properly color every planar graph.

$\psi \quad \phi \quad \chi \quad \pi$

A b c ... x X χ

Greek "chi"

Given a graph Γ, the chromatic number of Γ, denoted $\chi(\Gamma)$, is the smallest number of colors we can use to properly color the vertices of Γ. A proper coloring of the vertices of Γ is a coloring of the vertices such that no edge has both endpoints of the same color.

The theorem of Appel and Haken (1976) is that every planar graph Γ has $\chi(\Gamma) \le 4$.

Note that $\chi(K_n) = n$. Here K_n is the complete graph of order n.

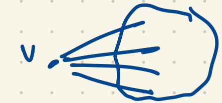
A graph Γ has $\chi(\Gamma) = 1$ iff it has vertices but no edges.

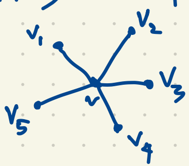
A graph Γ has $\chi(\Gamma) \le 2$ iff Γ is bipartite iff Γ has no circuits of odd length.

Computing $\chi(\Gamma)$ is hard in general.

Theorem If Γ is a finite planar graph then $\chi(\Gamma) \le 5$. Proof due to Heawood.

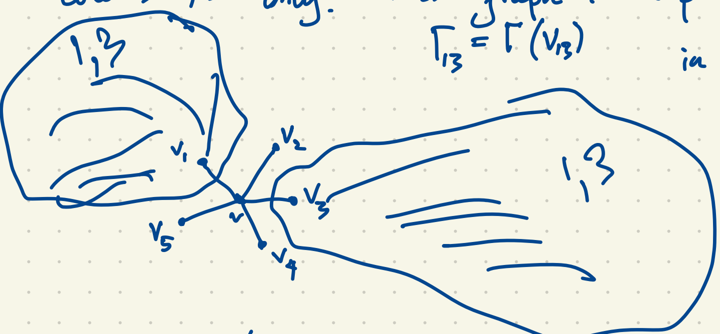
Proof If the theorem fails then there is a smallest counterexample Γ with n vertices (so Γ is planar and every planar graph of order $n-1$ has chromatic number ≤ 5 while $\chi(\Gamma) \ge 6$). We seek a contradiction. Γ has a vertex v of degree ≤ 5. In fact $\deg v = 5$. (If $\deg v \le 4$ then $\chi(\Gamma) \le 5$, a contradiction.) Let Γ' be the graph obtained



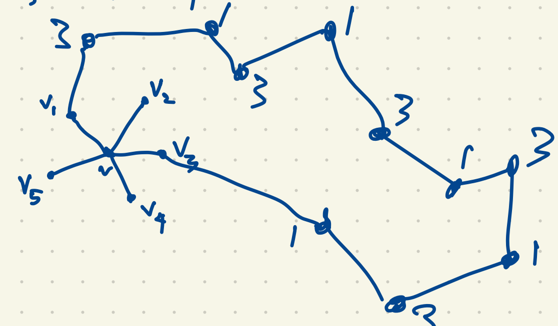


from Γ by deleting v and its five edges, so $\chi(\Gamma') \le 5$. Say v_i has color i $(i = 1, 2, \cdots, 5)$.

Consider the vertices $V_{13} \subset \{\text{vertices of } \Gamma\}$ having colors 1, 3 only. This graph is bipartite. $\Gamma_{13} = \Gamma(V_{13})$. I can assume v_1 is joined to v_3 in Γ_{13} (otherwise in part of Γ_{13}, reverse colors 1, 3 so that v_3 gets color 1. Then we are free to color v using color 3 since its neighbors are color 1, 2, 1, 4, 5).

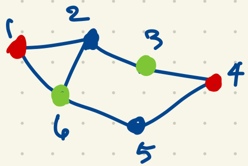


Otherwise Γ_{13} has a path from v_1 to v_3.

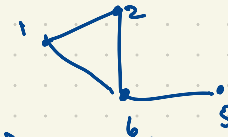


Similarly there is a path from v_2 to v_4 using only vertices of colors 2 and 4. Contradiction! $\square$

Given a graph Γ, a <u>subgraph</u> of Γ is formed by taking a subset of the edges of Γ together with all their vertices. An <u>induced subgraph</u> of Γ is formed by taking a subset of the vertices of Γ together with all their edges in Γ

Γ = 

eg.  is a subgraph of Γ. (not an induced subgraph of Γ)

eg.  is an induced subgraph of Γ.

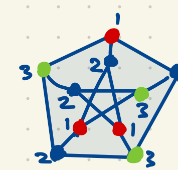
An induced subgraph of Γ is a subgraph of Γ, but not conversely.

A <u>k-clique</u> in Γ is a complete subgraph of Γ, i.e. a subset of the vertices, any two of which are joined.

In Γ above, $\{1,2,6\}$ is a clique (in fact a 3-clique). The <u>clique number</u> of Γ, denoted $\omega(\Gamma)$, is the size of the largest clique in Γ. It is hard to compute $\omega(\Gamma)$.

W vs. ω
Roman Greek

<u>Theorem</u> For every graph Γ, $\chi(\Gamma) \geq \omega(K)$.



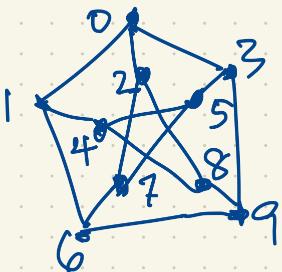
$\chi(P) = 3$ for the Petersen graph P.

Warning: this not equality! For the Petersen graph P, $\omega(P) = 2$.

Proof: The vertices in a clique of size $\omega(\Gamma)$ require $\omega(\Gamma)$ different colors.

Dual to the clique number $\omega(\Gamma)$ we have the coclique number $\alpha(\Gamma)$ which is the maximum number of vertices in Γ, no two of which are joined. (This is $\alpha(\Gamma) = \omega(\bar{\Gamma})$ where $\bar{\Gamma}$ is the complementary graph). Cocliques are also called independent sets of vertices.

Eg. $\alpha(P) = 4$. $|\chi(P)| \geq \frac{10}{4} = 2.5 \implies \chi(P) \geq 3$.



$\{1,2,3\}$ is a coclique which is not contained in any larger coclique; it is a <u>maximal</u> coclique.

A <u>maximum</u> coclique (i.e. a coclique of maximum size) is $\{1,3,7,8\}$.

This is maximum size because P has vertex set $\underline{\{0,3,9,6,1\} \cup \{2,7,5,4,8\}}$ as a union of two 5-cycles (circuits of length 5). Any set of size at least 5 vertices has either 3 on the inner 5-cycle $\{2,7,5,4,8\}$ or 3 vertices on the outer 5-cycle $\{0,3,9,6,1\}$. In either case there is an edge in that 5-cycle joining two vertices we have chosen.

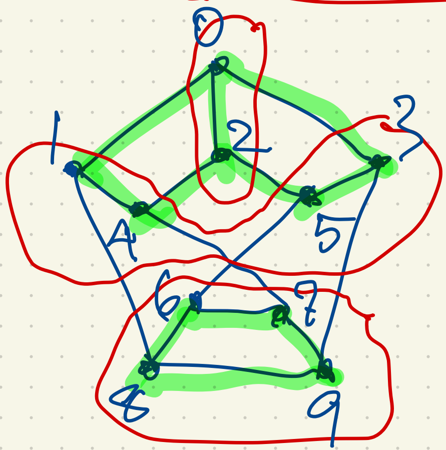
<u>Theorem</u> $\chi(\Gamma) \geq \dfrac{|V|}{\alpha(\Gamma)}$ where $|V|$ = the number of vertices = the order of Γ.

<u>Proof</u> Let $k = \chi(\Gamma)$. Properly color the vertices $1, 2, \cdots, k$ and let V_i be the subset of vertices colored i, for $i = 1, 2, \cdots, k$. This gives a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ ($V_i \cup V_j$ = union of V_i and V_j; $V_i \sqcup V_j$ = disjoint union of V_i and V_j). Each V_i is a coclique so $|V_i| \leq \alpha(\Gamma)$ so $|V| = |V_1| + |V_2| + \cdots + |V_k| \leq \underbrace{\alpha(\Gamma) + \alpha(\Gamma) + \cdots + \alpha(\Gamma)}_{k} = k\,\alpha(\Gamma)$ so $\chi(\Gamma) = k \geq \dfrac{|V|}{\alpha(\Gamma)}$. $\square$

March	M	W	F
		1	3
	6	$\boxed{8}$	10
	13	15	17

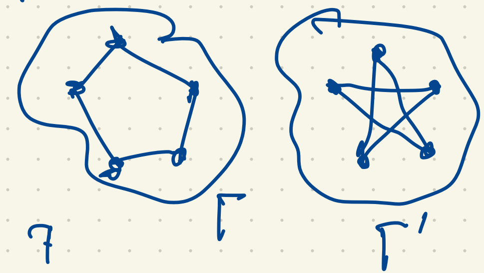
Spring Break

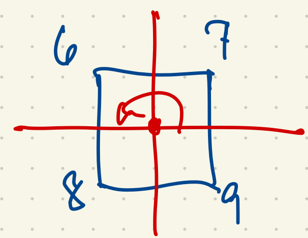
Test 1 : Wed Mar 8



$= \Gamma$

You can use nauty to test isomorphism between two graphs.





Using nauty Software

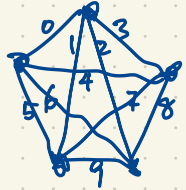
$G = \text{Aut}\,\Gamma$

$G = \langle (1,3)(4,5)(6,7)(8,9),\ (0,2)(1,4)(3,5)(6,9)(7,8) \rangle$

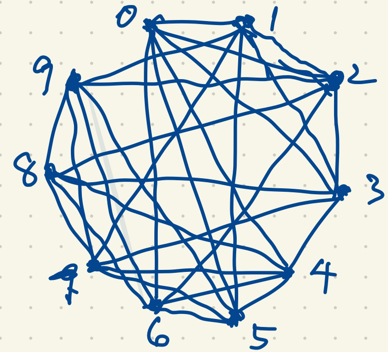
$|G| = 4$

G has 3 orbits on the vertices: $\{0,2\}, \{1,3,4,5\}, \{6,7,8,9\}$

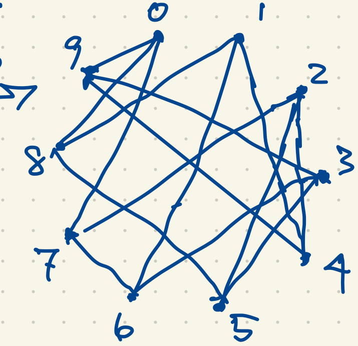
$K_5 =$



$\longrightarrow$ K_5'

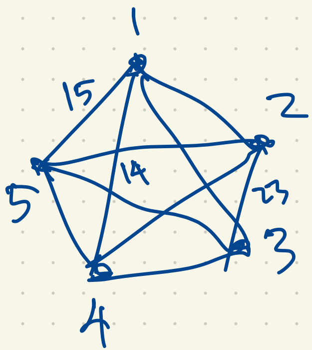


$\xrightarrow{\overline{K_5'}}$

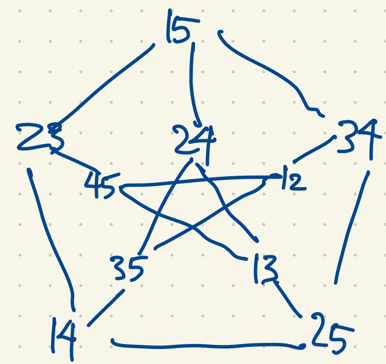


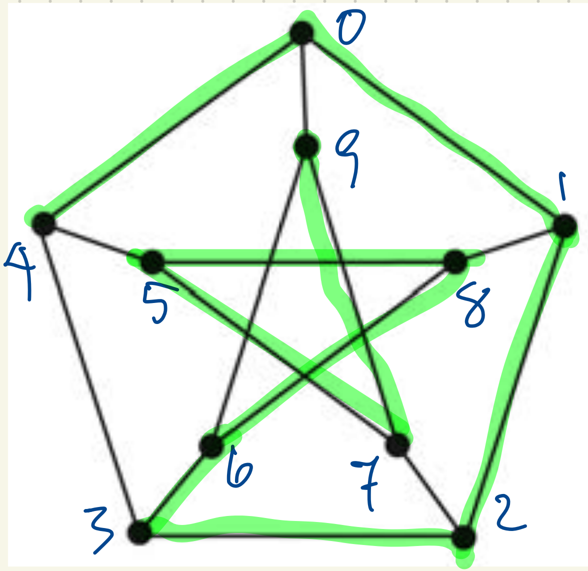
$|\mathrm{Aut}\ \overline{K_5'}| = 120$

$K_5 =$

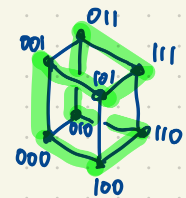


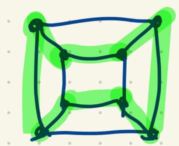
$\overline{K_5'} =$





The Petersen graph P has a Hamilton path $(0,1,2,3,6,8,5,7,9)$ (a path touching each vertex exactly once) but no Hamilton circuit (ending at the same vertex where it started).

The Hamming cube H_3 = 

= 

does have a Hamilton circuit.

000
100
110
010
011
111
101
001
000

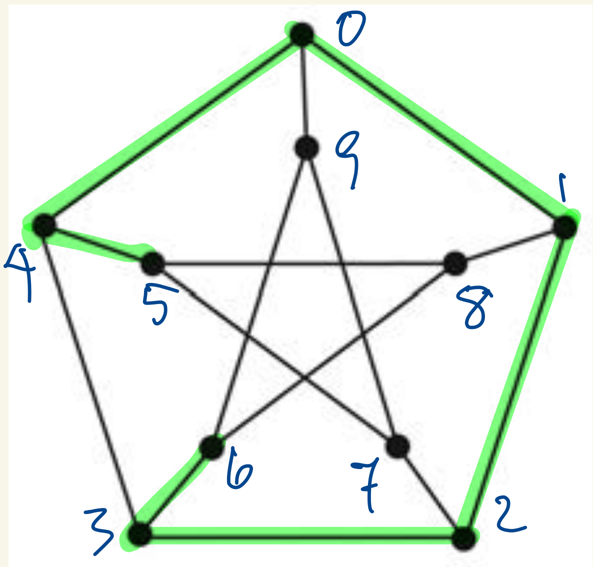
"Gray code"

Every Hamming graph H_n $(n \geq 2)$ has a Hamilton circuit.

A graph having a Hamilton circuit is called <u>Hamiltonian</u>.

Looking for Hamilton paths or circuits is known to be difficult in general.

Testing whether a giving graph Γ is Hamiltonian is NP-complete.

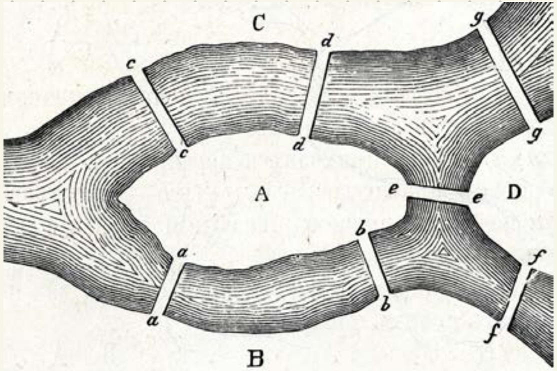


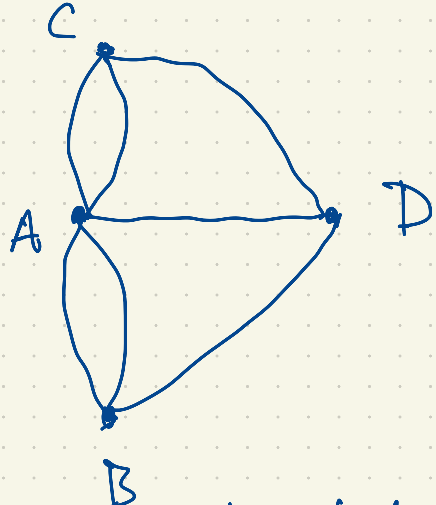
<u>Theorem</u>: The Petersen graph P is not Hamiltonian, i.e. it does not have a Hamilton circuit/cycle.

<u>Proof</u> Suppose P has a Hamilton circuit. Without loss of generality this circuit contains the path (4, 0, 1, 2). (This is because P has 120 automorphisms mapping any such path of length 3 to any other.) The Hamilton circuit uses two of the edges from vertex 3, so it uses either {3,4} or {2,3}; so without loss of generality, it uses the edge {2,3}. We cannot use the edge {3,4} as this would complete the circuit without passing through all vertices; so we must use the edges {3,6} and {4,5}. To continue the circuit from vertex 6, we have two choices: proceed through vertex 8 or vertex 9. Neither of these choices leads to a Hamilton circuit. This is a contradiction. □

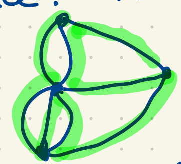
Euler paths and circuits

The Seven bridges of Königsberg





An <u>Euler trail</u> is a trail (repeating vertices but not edges) which uses each edge exactly once. An <u>Euler circuit</u> is an Euler trail that returns to its starting point.



This graph has an Euler trail. In order to have an Euler trail, a graph must have either 0 or 2 vertices of odd degree. When there are no vertices of odd degree, we have an Euler circuit.

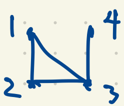
<u>Theorem</u> (Euler) A graph has an Euler trail iff it is connected and it has either 0 or 2 vertices of odd degree. In the case every vertex has even degree, we have an Euler circuit/cycle.

We sometimes speak of labelled graphs and unlabelled graphs.

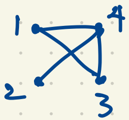
Eg. on the vertex set $\{1,2,3,4\}$, there are $2^6 = 64$ labelled graphs

$\binom{4}{2} = 6$ pairs of vertices.

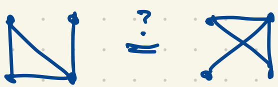
There are $\binom{n}{2}$ labelled graphs on n vertices.
But many of them are isomorphic.



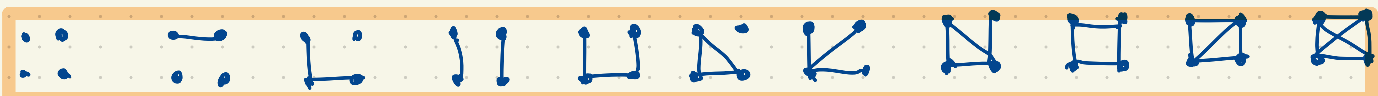
$\neq$



These are different labelled graphs but they are isomorphic.



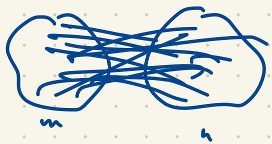
As unlabelled graphs they are isomorphic, hence the same graph.



There are 11 unlabelled graphs of order 4 i.e. 11 isomorphism types of graphs of order 4, i.e. 11 graphs of order 4 up to isomorphism.

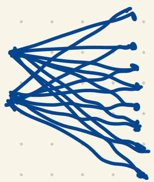
The Petersen graph has girth 5 (the shortest cycle has length 5). It has 15 edges.
For a graph on 10 vertices, 15 edges is the maximum possible for girth 5.
For a graph on 10 vertices without triangles (i.e. girth ≥ 4), what is the maximum possible number of edges?

Recall: $K_{m,n}$ =



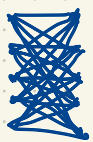
$K_{m,n}$ has mn edges. $K_{m,n}$ is bipartite so it has no cycles of odd length.
In particular it has no triangles.

$K_{2,8}$ has 16 edges



$K_{2,8}$

16 edges



$K_{5,5}$

25 edges
girth 4
(no triangles)

Theorem (Mantel 1907)
If Γ is a graph of order n with no triangles (i.e. its girth is at least 4) then Γ has at most $\frac{n^2}{4}$ edges.

If n is even then $K_{\frac{n}{2},\frac{n}{2}}$ attains the upper bound of $\frac{n^2}{4}$ edges. What if n is odd?

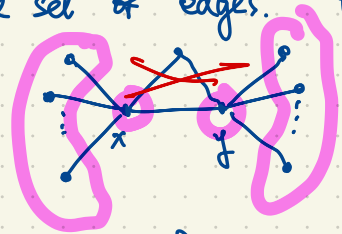
$$\left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even, for } K_{\frac{n}{2},\frac{n}{2}} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd, for } K_{\frac{n+1}{2},\frac{n-1}{2}} \end{cases}$$

On 9 vertices, any graph without triangles has at most $\underline{20}$ edges.

$K_{4,5}$

__Proof__ Let Γ be a graph of order n with no triangles, $\Gamma = (V, E)$. (V is the set of vertices, E is the set of edges. For every edge $\{x,y\} \in E$, $d(x) + d(y) \leq n$.

$d(x)-1$

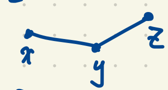


$d(y)-1$

$$d(x) - \cancel{1} + \cancel{1} + \cancel{1} + d(y) - \cancel{1} \leq n$$

Add the inequality $d(x)+d(y) \leq n$ over all edges $\{x,y\} \in E$ to get $\sum_{\{x,y\} \in E} (d(x)+d(y)) \leq ne$.

Next, count the number of triples of vertices (x,y,z) with $x \sim y \sim z$.

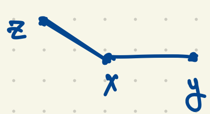


There are n choices for $y \in V$ and $d(y)$ choices for x, $d(y)$ choices for z, so $d(y)^2$ choices for x and z (given y). The total number of walks of length 2 is $\sum_{y \in V} d(y)^2$.

On the other hand, there are $e = |E|$ edges in Γ. For the edge $\{x,y\} \in E$, how many walks of length 2 contain this edge? $d(x)+d(y)$ choices of walk of length 2 in which we include a step from x to y

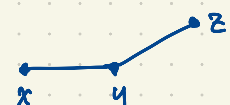
The total number of walks of length 2 is

$$\sum_{\{x,y\} \in E} (d(x)+d(y)).$$



$d(x)$ choices for z
(given the edge $\{x,y\}$)

or



$d(y)$ choices for z
(given the edge $\{x,y\}$)

Therefore

$$\sum_{x\in V} d(x)^2 = \sum_{\{x,y\}\in E} \underbrace{(d(x)+d(y))}_{\leq n} \leq en$$

If $\sum_{x\in V} d(x) = 2e$, what does this tell us about $\sum_{x\in V} d(x)^2$?

Use the Cauchy-Schwarz inequality.

$$(\underbrace{d(1)+\cdots+d(n)}_{2e})^2 \leq n(d(1)^2+\cdots+d(n)^2)$$

$$4e^2 \leq n\sum_{x\in V} d(x)^2 = n\sum_{\{x,y\}\in E}(d(x)+d(y)) \leq n\cdot ne = n^2 e$$

So $e \leq \frac{n^2}{4}$.

Cauchy-Schwarz Inequality

Given two vectors

$a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

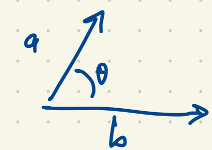
$b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$,

$$|a\cdot b| \leq \|a\|\cdot\|b\|$$

where $a\cdot b = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$$a\cdot b = \|a\|\,\|b\|\cos\theta$$



$|\cos\theta| \leq 1$.

Special case: $b = (1, 1, 1, \cdots, 1)$

$\|b\| = \sqrt{\underbrace{1^2+1^2+\cdots+1^2}_{n}} = \sqrt{n}$

$a\cdot b = a_1 + a_2 + \cdots + a_n$

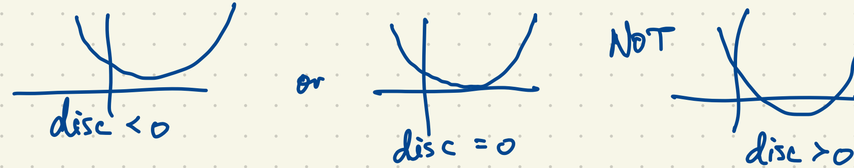
$|a\cdot b| \leq \|a\|\cdot\sqrt{n}$

$$(a_1+\cdots+a_n)^2 \leq n(a_1^2+\cdots+a_n^2)$$

Proof of Cauchy-Schwarz: Fix $a, b \in \mathbb{R}^n$. Consider

$$f(t) = \|a - tb\|^2 = (a-tb)\cdot(a-tb) = a\cdot a - t\,b\cdot a - t\,a\cdot b + t^2\,b\cdot b = \|a\|^2 - 2t(a\cdot b) + t^2\|b\|^2.$$

Of course $f(t) \geq 0$ for all t.



The discriminant

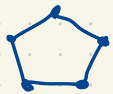
$$(-2a\cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$

$$4(a\cdot b)^2 \leq 4\|a\|^2\|b\|^2$$

$$(a\cdot b)^2 \leq \|a\|^2\|b\|^2.$$

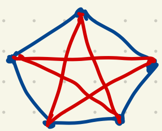
The disc. of at^2+bt+c is b^2-4ac

Show me a graph Γ of order 5 such that neither Γ nor $\bar{\Gamma}$ has a triangle.
(i.e. both Γ and $\bar{\Gamma}$ have girth ≥ 4). Recall: $\bar{\Gamma}$ is the complement of Γ.

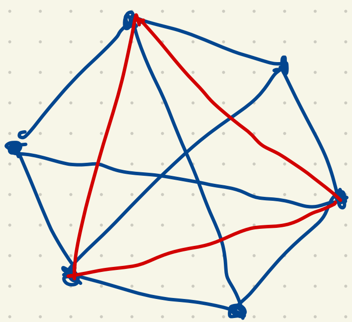
$\Gamma =$  $\bar{\Gamma} =$  $\cong \Gamma$ (a 5-cycle). Both Γ and $\bar{\Gamma}$ have girth 5.

Show me a graph Γ of order 6 such that neither Γ nor $\bar{\Gamma}$ have a triangle.
There is no such graph. Why not?

Color the edges of K_5 with 2 colors red, blue. To avoid a monochromatic triangle (all red or all blue):

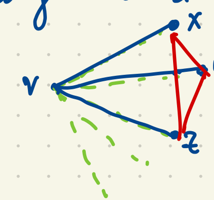


K_6:



Theorem If we color the edges of K_6 red and blue, then there is either a red triangle or a blue triangle.

Proof Consider a vertex v.



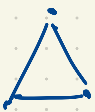
There are five edges from v so by the Pigeon hole Principle, at least three of them are the same color, say $\{v,x\}$, $\{v,y\}$, $\{v,z\}$ are blue.

Now x, y, z form a triangle.
If any edge of this triangle is blue then together with the edges to v we have a blue triangle. Otherwise all edges of the triangle $\{x,y,z\}$ are red. $\square$

Theorem (Ramsey) Let r, s be positive integers. There is a number $R(r,s)$ such that for all $n \geq R(r,s)$, every 2-coloring (red, blue) of the edges of K_n has either a red r-clique or a blue s-clique. For $n < R(r,s)$, there exists a coloring of the edges of K_n with 2 colors (red, blue) having no red r-clique and no blue s-clique.

$R(3,3) = 6.$

$R(2,3) = 3$



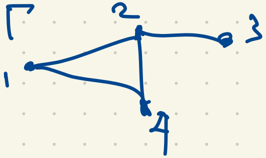
$43 \leq R(5,5) \leq 48$

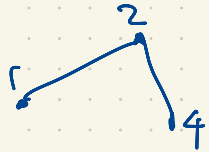
$R(r,s)$ values are Ramsey numbers.

(Extremal graph theory)

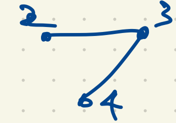
Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[9]	36–40	49–58	59[13]–79	73–106	92–136
5					**43–48**	58–85	80–133	101–194	133–282	149[13]–381
6						**102–161**	115[13]–273	134[13]–427	183–656	204–949
7							**205–497**	219–840	252–1379	292–2134
8								**282–1532**	329–2683	343–4432
9									**565–6588**	581–12677
10										**798–23556**

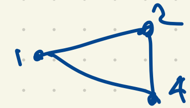




is a subgraph of Γ



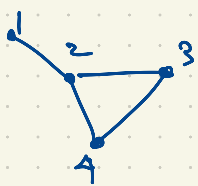
is not a subgraph.



is an induced subgraph.

$$\underbrace{\text{Graph Theory} \longleftrightarrow \text{Linear Algebra}}_{\text{Matroid Theory}}$$

The adjacency matrix of a graph Γ with vertices $1, 2, 3, \dots, n$ is the $n \times n$ matrix whose (i,j)-entry is the number of edges from vertex i to vertex j. Eg.

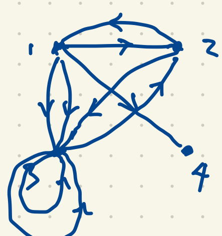
$\Gamma =$ 

has adjacency matrix

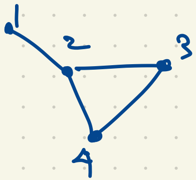
$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = A$$

Symmetric $(0,1)$-matrix with "zero diagonal" (corresponding to an undirected graph with no loops or multiple edges)

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

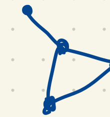


directed graph with loops and multiple edges

$\Gamma =$ 

has adjacency matrix

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array} = A$$

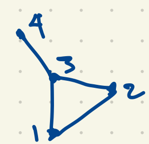
The unlabelled graph  has several choices of adjacency matrix.

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 3 & 1 & 1 \\ 3 & 1 & 1 & 2 & 1 \\ 4 & 1 & 1 & 1 & 2 \end{array}$$

The (i,j)-entry of A^2 is the number of walks of length 2 from vertex i to vertex j.
(A walk is allowed to repeat edges or vertices, unlike in a path or a trail.)

$$A^3 = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}}_{A^2}\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{bmatrix}$$

The (i,j)-entry of A^m is the number of walks of length m from vertex i to vertex j in Γ.



has adjacency matrix $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$A^0 = I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^1 = A$$

$$J = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad j = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

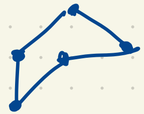
If Γ is a k-regular graph then $Aj = kj$, $AJ = kJ$
i.e. j is an eigenvector of A with eigenvalue k.

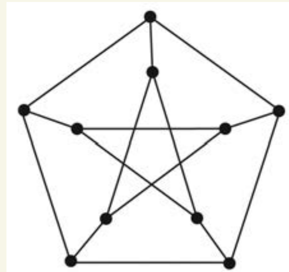
$$Aj = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$ degree sequence

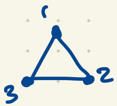
$$AJ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$$

Which graphs are connected regular with diameter 2 and girth 5?

Any such graph has order 5, 10, 50 or 3250.

5-cycle 

Petersen graph 

Look at $K_3 =$ 

with adjacency matrix $A = \begin{array}{c|c} i \backslash j & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$.

The number of walks of length k from vertex i to vertex j is the (i,j)-entry of A^k.

How many walks of length 10 are there from vertex 2 to vertex 3 in K_3? 341.

$$A^{10} = \begin{bmatrix} 342 & 341 & 341 \\ 341 & 342 & 341 \\ 341 & 341 & 342 \end{bmatrix}$$

Since A is a real symmetric matrix, it is diagonalizable i.e. there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. One obvious eigenvector $\underline{u}_1 = \underline{j} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$: $A\underline{j} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\underline{j}$

since K_3 is 2-regular. Another eigenvector $\underline{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$: $A\underline{u}_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = -\underline{u}_2$

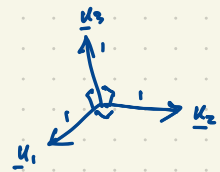
$\underline{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $A\underline{u}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = -\underline{u}_3$

It is customary to normalize the eigenvectors as

$$\underline{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \underline{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \underline{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

so that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

Orthonormal means $\underline{u}_i \cdot \underline{u}_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$



$A\underline{u}_i = \lambda_i \underline{u}_i \qquad \lambda_1 = 2, \quad \lambda_2 = \lambda_3 = -1.$

A has diagonal form $D = \begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$. $\quad D^{10} = \begin{bmatrix}1024&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$A = UDU^{-1}$ i.e. $U^{-1}AU = D$ where $U = \begin{bmatrix}\underline{u}_1 & \underline{u}_2 & \underline{u}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}}\end{bmatrix}$

$$A^{10} = (UD\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})(\cancel{U}D\cancel{U^{-1}})\cdots(\cancel{U}DU^{-1}) = UD^{10}U^{-1} = \begin{bmatrix}342&341&341\\341&342&341\\341&341&342\end{bmatrix}$$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ has eigenvector $\begin{bmatrix}1\\0\end{bmatrix}$ but no basis of eigenvectors.

$\begin{bmatrix}3&1\\0&2\end{bmatrix}$ has a basis of eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ (with eigenvalue 3) $\begin{bmatrix}1\\-1\end{bmatrix}$ (with eigenvalue 2) but they aren't perpendicular/orthogonal.