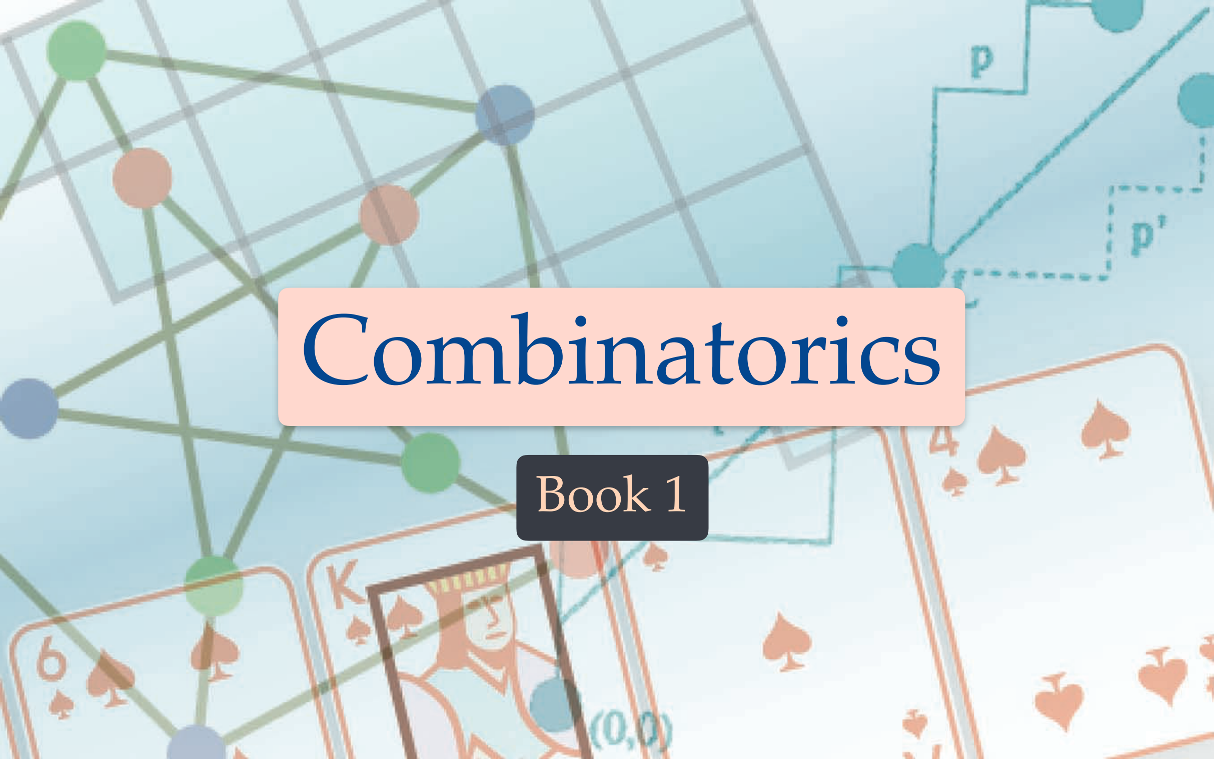
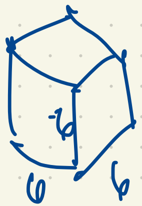


Combinatorics

Book 1

Eric

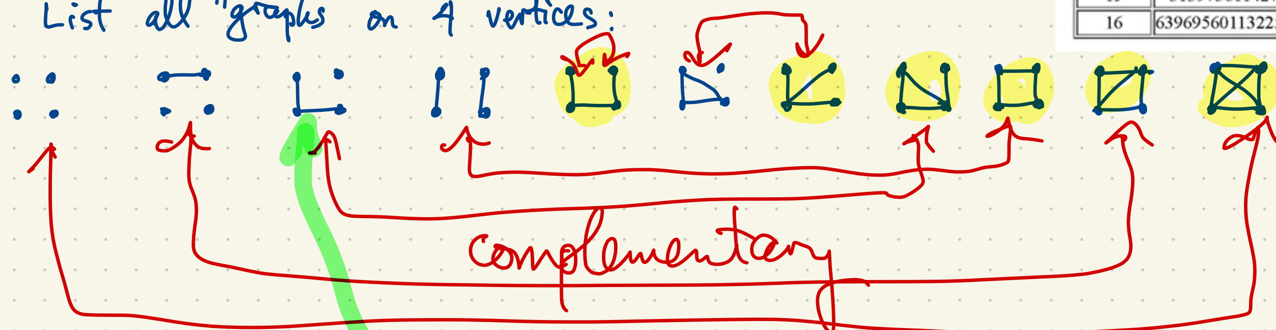
RH 

$6^3 = 216$

#vertices	Connected graphs	All graphs
1	1	1
2	1	2
3	2	4
4	6	11
5	21	34
6	112	156
7	853	1044
8	11117	12346
9	261080	274668
10	11716571	12005168
11	1006700565	1018997864
12	164059830476	165091172592
13	50335907869219	50502031367952
14	29003487462848061	29054155657235488
15	31397381142761241960	31426485969804308768
16	63969560113225176176277	64001015704527557894928

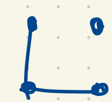
Ordinary / Simple Graph on n vertices/nodes

Eg. List all 11 graphs on 4 vertices:

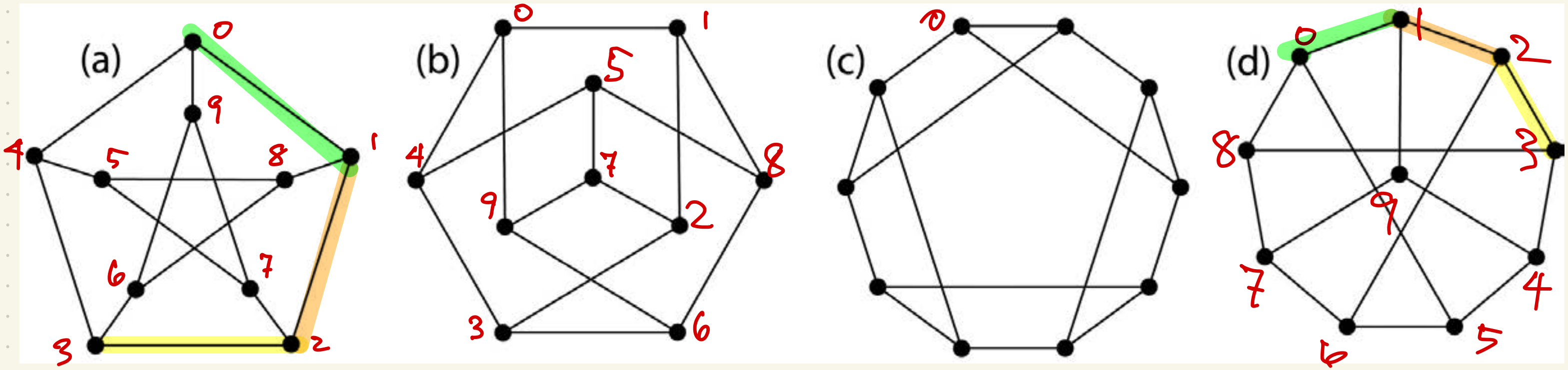


A graph of order n is a pair $G = (V, E)$ where V is a set of n vertices and E is a subset of pairs $\{v, w\}$ where $v \neq w$, $v, w \in V$. Eg. the graph with vertices 1, 2, 3, 4 and edges $\{1,3\}$, $\{2,3\}$ can be illustrated

$\cong$



(the two graphs are isomorphic).

Of these four graphs, which one is not isomorphic to the others?

Graphs (a), (b) are isomorphic. Graph (c) is not isomorphic to (a) or (b) because graph (a) has diameter 2: any two vertices are at distance at most 2 apart. However, graph (c) has diameter 3.

An automorphism (symmetry) of a graph is an isomorphism from the graph to itself.

An isomorphism from graph (a) to graph (d) is the map with table of values

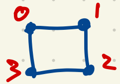
vertex in (a)	vertex in (d)
0	0
1	1
2	2
3	3
4	8
5	7
6	4
7	6
8	9
9	5

This is a very special graph having the special property that for every path of length 3 (vertices v_0, v_1, v_2, v_3 with $v_0 \sim v_1 \sim v_2 \sim v_3$, $v_0 \not\sim v_2$, $v_0 \not\sim v_3$, $v_1 \not\sim v_3$) in (a) and every path $w_0 \sim w_1 \sim w_2 \sim w_3$ in (d) ($w_0 \not\sim w_2$, $w_0 \not\sim w_3$, $w_1 \not\sim w_3$) there is a unique isomorphism (a) $\to$ (b) mapping $v_i \mapsto w_i$.

This is a Petersen graph. How many isomorphisms are there from (a) to (d)?

$10 \times 3 \times 2 \times 2 = 120.$

In particular, a Petersen graph has 120 automorphisms.

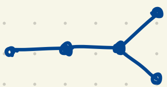
The graph  (a 4-cycle) has 8 automorphisms

$0 \mapsto 1$	$0 \mapsto 0$	$0 \mapsto 0$	
$1 \mapsto 2$	$1 \mapsto 3$	$1 \mapsto 1$	
$2 \mapsto 3$	$2 \mapsto 2$	$2 \mapsto 2$	
$3 \mapsto 0$	$3 \mapsto 1$	$3 \mapsto 3$	
		identity	

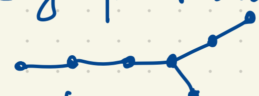
Not an automorphism:

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto 3$
$3 \mapsto 2$

The edge $0 \sim 3$ is mapped to a non-edge $0 \nsim 2$

The graph  has exactly 2 automorphisms

A graph with only one automorphism? • (the graph of order 1, i.e. having only one vertex).

A less trivial example with more than one vertex:  Γ

Every graph as a <u>degree sequence</u>. The degree of a vertex is the number of its neighbors.

The graph Γ (above) has degree sequence $(1,1,1,2,2,2,3)$. $1+1+1+2+2+2+3=12$

If two graphs are isomorphic, they must have the same degree sequence.

An isomorphism from Γ to Γ' must map each vertex to a vertex of the same degree.

If two graphs have the same degree sequence, must they be isomorphic? No, e.g. the graphs (a), (c) on the previous page are not isomorphic, but both have degree sequence $(3,3,3,3,3,3,3,3,3,3)$.

A graph with n vertices and e edges has <u>order</u> n. The <u>degree</u> of vertex v, denoted $\deg(v)$, is the number of vertices joined to v. If G has vertices labelled $1,2,3,\dots,n$, then the degree sequence of G is $(\deg(1), \deg(2), \cdots, \deg(n))$, permuted into increasing order. A graph G is <u>d-regular</u> if $\deg(v)=d$ for every vertex v in G (or simply regular). Note: $\deg(1)+\deg(2)+\cdots+\deg(n) = 2e$.

Theorem If G is a (finite) simple graph with e edges, then $\sum_{v\in V}\deg(v) = 2e$ where $G=(V,E)$, V the set of vertices, E the set of edges.

Proof We count in two different ways the number of pairs $(v, \{v,w\})$ in G ($v\in V$, $\{v,w\}\in E$).

Since every edge $\{v,w\}$ has two vertices v, w, there are $2e$ such pairs.

On the other hand, since each vertex $v\in V$ has $\deg(v)$ edges, we have $\sum_{v\in V}\deg(v)$ as the number of such pairs. These answers must agree. $\square$

Imagine we organize a round robin fencing tournament between n competitors. Every competitor competes with each of the others exactly once. Altogether there are $\binom{n}{2} = \frac{n(n-1)}{2}$.

In general $\binom{n}{k}$ = "n choose k" is the number of ways to choose a k-subset of an n-set (i.e. a subset of size k in a set of n elements). $\binom{n}{k}$ is a binomial coefficient.

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$ (the Binomial Theorem)

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b) = aaaaa + aaaab + aaaba + aaabb + \cdots + bbbbb$$

Before collecting terms, there are $2^5 = 32$ terms.

$$= \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b^1 + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}a^1b^4 + \binom{5}{5}a^0b^5$$

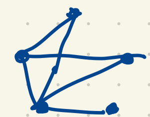
$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Theorem In a simple graph with $n \geq 2$ vertices, there exist two vertices of the same degree.

Proof Let $(d_1, d_2, \cdots, d_n)$ be the degree sequence of a graph of order $n \geq 2$. Note that $d_1, \cdots, d_n \in \{0, 1, 2, \cdots, n-1\}$. If $d_1, \cdots, d_n$ are distinct then every element of $\{0, 1, 2, \cdots, n-1\}$ is the degree of some vertex by the Pigeonhole Principle. This means the degree sequence is $(0, 1, 2, \cdots, n-1)$. In particular, there is a vertex of degree 0, and a vertex of degree $n-1$, a contradiction. This proves the result. $\square$



degrees $1, 2, 2, 3, 4$

Note: $i \mapsto d_i$

$\{1, 2, \cdots, n\} \to \{0, 1, 2, \cdots, n-1\}$.

Proofs are logical arguments that argue the truth of our assertion. They are always written in proper sentences.

singular	plural
vertex	vertices
index	indices
matrix	matrices

has degree sequence $(0, 1, 1)$.

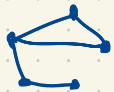
$\{0, 1\}$ is the set of degrees of the vertices

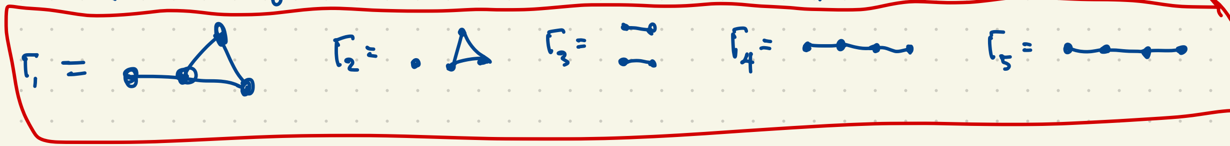
Pigeonhole Principle Suppose n pigeons come to roost in k holes. If $n > k$, then two (at least) pigeons must be in the same hole. If $n < k$, at least one of the holes will be empty. In other words, if $f: A \to B$ is any function where $|A| = n$ and $|B| = k$, then: (i) If $n > k$ then f cannot be one-to-one; (ii) if $n < k$ then f cannot be onto. (iii) Assuming $n = k$ then f is one-to-one iff it is onto.

Graph Reconstruction Problem

actually multiset

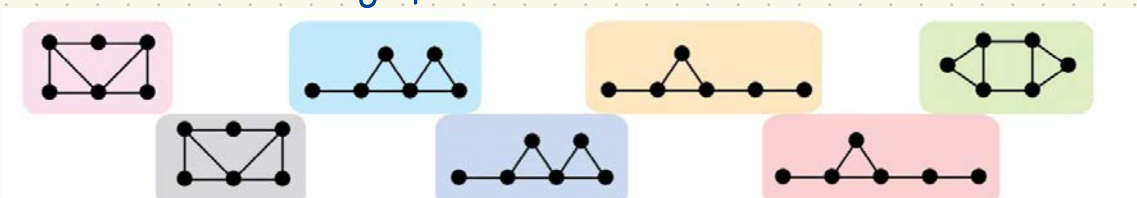
Starting with a (simple) graph Γ of order n, we construct a set of n graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ where Γ_i is formed by deleting vertex i (and all edges from vertex i). The set $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ is called the *deck* of Γ.

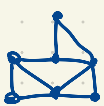
e.g. $\Gamma =$ 



Can you (uniquely) reconstruct Γ from its deck?

Consider this (multiset) set of seven graphs of order 6. Find a graph Γ of order 7 having this as its deck.



Answer: 

Note: From the deck of any graph Γ, we can reconstruct (deduce) the degree sequence of Γ.

Given two graphs of order n, how hard is it to check whether they are isomorphic?

Assuming Γ, Γ' are given, each with n vertices, label the vertices of each graph $1, 2, 3, \cdots, n$. The number of bijections from the vertices of Γ to the vertices of Γ' is $n! = 1 \times 2 \times 3 \times \cdots \times n$ (n factorial). (eg. $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $\cdots$, $10! = 3628800$, $\cdots$). Check each of the bijections to see if it is an isomorphism. This takes at most $n!\binom{n}{2}$.

We have an algorithm for testing graph isomorphism but it requires (in the worst case) $n!\binom{n}{2}$ steps where n is the order of the graphs.

$n! \to \infty$ faster than any polynomial in n i.e. if $f(n)$ is a polynomial in n (eg. $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$ where k is constant... $\binom{n}{k}$ is a polynomial of degree k in n.)

i.e. $\lim_{n\to\infty} \frac{n!}{f(n)} = \infty$ for any positive polynomial function $f(n)$.

In fact, $n! \to \infty$ faster than any exponential function c^n $(c>1)$ eg.

$$\lim_{n\to\infty} \frac{n!}{10^n} = \lim_{n\to\infty} \left(\frac{1}{10}\cdot\frac{2}{10}\cdot\frac{3}{10}\cdots\frac{9}{10}\cdot\frac{10}{10}\cdot\frac{11}{10}\cdot\frac{12}{10}\cdot\frac{13}{10}\cdots\cdot\frac{n}{10}\right) = \infty$$

The best algorithms known ~~for testing~~ for graph isomorphism require far fewer than $n!\binom{n}{2}$ steps (even in the worst case). These algorithms have running time that is intermediate between polynomial and exponential.

In the worst case, it takes $O(n^2)$ steps to compute the degree sequence of a graph, a polynomial function of n.

Assume graph P (the Petersen graph) has 120 automorphisms. ($P \cong$ graph (a))

If Γ is any graph, then either $P \not\cong \Gamma$ or there are exactly 120 isomorphisms $P \to \Gamma$.

If $f : \underbrace{V(P)}_{\text{vertices of } P} \to \underbrace{V(\Gamma)}_{\text{vertices of } \Gamma}$ is an isomorphism then for every automorphism $\theta : V(P) \to V(P)$, we have an isomorphism

$$\begin{tikzcd} V(P) \arrow[r, "\theta"] \arrow[rr, bend right, "f\circ\theta"'] & V(P) \arrow[r, "f"] & V(\Gamma) \end{tikzcd}$$

Given two graphs Γ, Γ', there may be no isomorphism from Γ to Γ'. But if there is an isomorphism $f : \Gamma \to \Gamma'$, then the number of isomorphisms $\Gamma \to \Gamma'$ is equal to the number of automorphisms of Γ:

$$\mathrm{Aut}(\Gamma) = \{\text{automorphisms of } \Gamma\} \xleftrightarrow{1:1} \{\text{isomorphisms } \Gamma \to \Gamma'\}.$$

Given $\theta \in \mathrm{Aut}\,\Gamma$,

$$\begin{tikzcd} \Gamma \arrow[r, "\theta"] \arrow[rr, bend right, "f\circ\theta"'] & \Gamma \arrow[r, "f"] & \Gamma' \end{tikzcd}$$

$f\circ\theta : \Gamma \to \Gamma'$ is an isomorphism.

The map $\theta \mapsto f\circ\theta$ is one-to-one. Given any isomorphism $g : \Gamma \to \Gamma'$, there exists $\theta \in \mathrm{Aut}\,\Gamma$ such that $g = f\circ\theta$. Why?

$$\begin{tikzcd} \Gamma \arrow[r, "g"] \arrow[rr, bend right] & \Gamma' \arrow[r, "f^{-1}"] & \Gamma \end{tikzcd}$$

$f^{-1}\circ g : \Gamma \to \Gamma$ is an isomorphism

i.e. $\theta = f^{-1}\circ g \in \mathrm{Aut}\,\Gamma$

and $f\circ\theta = f\circ(f^{-1}\circ g) = \underbrace{(f\circ f^{-1})}_{\iota}\circ g = g$.

$\mathrm{Aut}\,\Gamma$ is a group:

$\iota : \Gamma \to \Gamma$ identity

$\iota(x) = x$ for every vertex x of Γ

$\iota \in \mathrm{Aut}\,\Gamma$

$\iota\circ\theta = \theta = \theta\circ\iota$ for all $\theta \in \mathrm{Aut}\,\Gamma$;

$(\rho\circ\sigma)\circ\tau = \rho\circ(\sigma\circ\tau)$ for all $\rho, \sigma, \tau \in \mathrm{Aut}\,\Gamma$;

for every $\theta \in \mathrm{Aut}\,\Gamma$ there exists $\theta^{-1} \in \mathrm{Aut}\,\Gamma$ such that $\theta\circ\theta^{-1} = \iota = \theta^{-1}\circ\theta$.

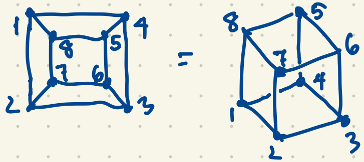
$n!$ grows faster than c^n for any $c > 1$; $n!$ grows slower than n^n:
(exponential)

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim \Big(\frac{1}{n}\cdot\underbrace{\frac{2}{n}\cdot\frac{3}{n}\cdot\frac{4}{n}\cdot\ \cdots\ \cdot\frac{n-1}{n}\cdot\frac{n}{n}}_{<\frac{1}{n}}\Big) = 0$$

<u>Stirling's Approximation</u>

$n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ as $n \to \infty$ i.e. $\displaystyle\lim_{n\to\infty} \frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}} = 1.$

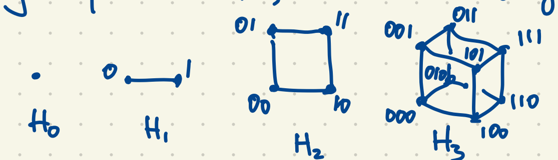
asymptotic

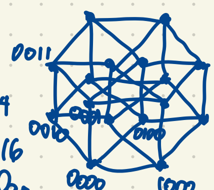
Cube: $\Gamma =$ 

How many automorphisms does Γ have?

The cube has 48 symmetries (24 rotational and 24 other)

Hamming Cube graph H_n has as its vertices all bitstrings of length n. A <u>bit</u> is a 0 or a 1 in the binary alphabet $\{0,1\}$. Two bitstrings are adjacent if they differ in exactly one position. H_n is a regular graph degree n with 2^n vertices.





H_4
order 16
4-regular

H_n has diameter n and $|\mathrm{Aut}\, H_n| = 2^n n!$

In H_4, switch 2nd and 4th coordinates.

$d'(0011, 1011) = d(0110, 1110) = 1$

The $n!$ permutations of the coordinates of the strings give $n!$ automorphisms of H_n.

Also there are 2^n possible bit flip operations (flip a single coordinate $0 \leftrightarrow 1$, or do this for a subset of the coordinates).

Eg. Flip the 2nd and 3rd bits:

$d(0011, 1011) = d(0101, 1101) = 1$

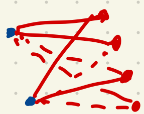
eg. $|\mathrm{Aut}\, H_3| = 2^3 \cdot 3! = 8 \times 6 = 48$ as above.

Some infinite graphs:  2-regular, connected

H_∞ has as its vertices the bitstrings $a_1a_2a_3a_4\cdots$, $a_i \in \{0,1\}$ e.g.

H_∞ is not connected.

Random infinite graphs are connected. They have diameter 2.



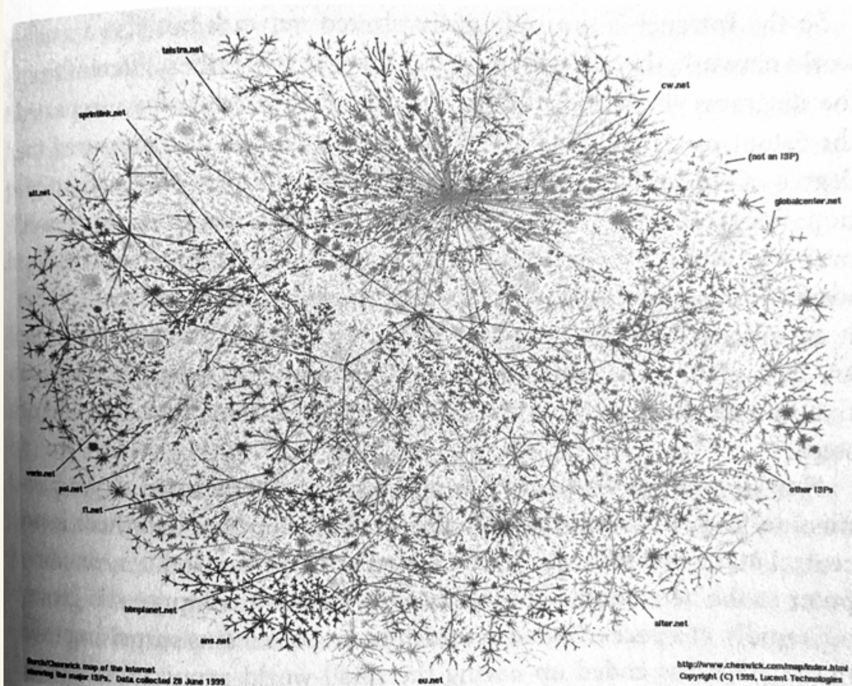
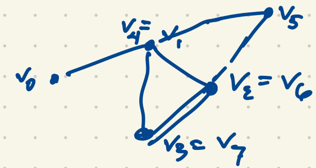


Figure 8. A map of the Internet. (Reprinted by permission of Bill Cheswick and Lucent Technologies.)

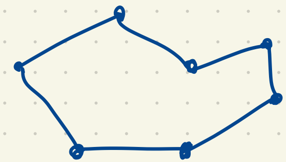
A *walk* in a graph Γ is a sequence of vertices $(v_0, v_1, v_2, \cdots, v_r)$ such that v_i is adjacent to v_{i+1} for $i = 0, 1, 2, \cdots, r-1$. This walk has length r.



A *path* is a walk without repeating vertices.

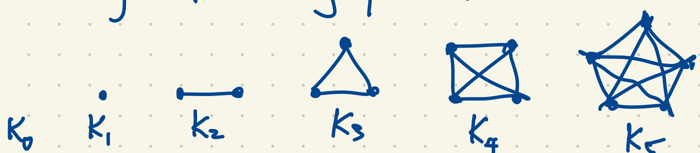
A *trail* is a walk which possibly repeats vertices but does not repeat edges.

A *circuit* is a sequence of vertices $(v_0, v_1, v_2, \cdots, v_r)$ such that $v_i \sim v_{i+1}$ for $i = 0, 1, 2, \cdots, r-1$ where $v_0, v_1, \cdots, v_r$ are distinct except $v_0 = v_r$.



length 7.

The complete graph K_n is the graph of order n in which $x \sim y$ for every pair of distinct vertices x, y.

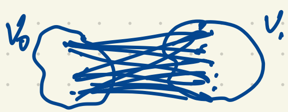


etc.

Note: K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.

The *complete bipartite graph* $K_{m,n}$ is the graph of order $m+n$ and mn edges having vertex set $V_0 \sqcup V_1$ ($A \cup B$ is the union of two sets, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

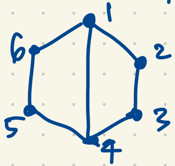
$A \sqcup B$ is the disjoint union where $A \cap B = \emptyset$ i.e. A, B are disjoint)



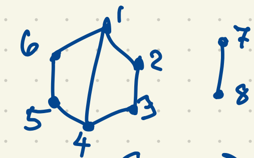
Given $x, y \in V_0 \sqcup V_1$, we have $x \sim y$ iff one of x, y is in V_0 and the other is in V_1.

Here $|V_0| = m$, $|V_1| = n$.

A <u>bipartite graph</u> has vertex set $V_0 \sqcup V_1$ and every edge has one endpoint in V_0, the other endpoint in V_1.

Eg. 

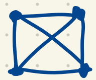
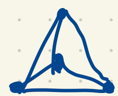
is bipartite with partition $\{1,3,5\} \sqcup \{2,4,6\}$.

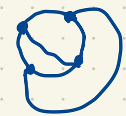


$\{1,3,5,7\} \sqcup \{2,4,6,8\}$ or $\{1,3,5,8\} \sqcup \{2,4,6,7\}$

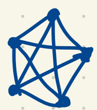
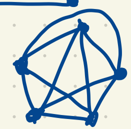
<u>Theorem</u> A graph is bipartite iff it has no circuits of odd length.

A <u>planar graph</u> is a graph which can be drawn in the plane $\mathbb{R}^2$ without crossing edges.

eg. K_4 =  =  is planar.

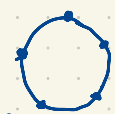


via the circle-chord method. This method works well when the graph has a <u>Hamilton circuit</u>: a circuit passing through each vertex once.

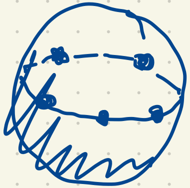
eg. K_5 =  =  is not planar.

<u>Theorem</u> K_5 is not planar.

<u>Proof</u> (one way) We must start with a Hamilton circuit of length 5



and find a way to add the remaining 5 edges without crossings. By the Pigeonhole Principle, we must add at least 3 edges inside the circle or at least 3 edges outside the circle. Without loss of generality, three (or more) edges are to be added inside the circle. But it is easy to see that any three such chords must have a point of crossing. So K_5 is not planar. □



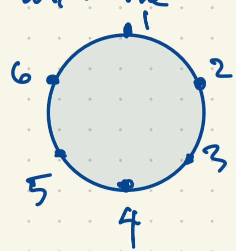
Stereographic Projection

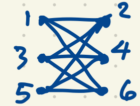
$\mathbb{R}^2 \cup \{N\} \longleftrightarrow S^2$ (sphere)

N: north pole

<u>Theorem</u> $K_{3,3}$ is not planar.

<u>Proof</u> Use the circle-chord method with the Hamilton circuit $(1,2,3,4,5,6,1)$. We must add the missing 3 edges. Without loss of generality, we add at least two edges inside the circle. But the only possible edges are $\{1,4\}$, $\{2,5\}$, $\{3,6\}$ but any two of these chords intersect. So $K_{3,3}$ is not planar. $\square$



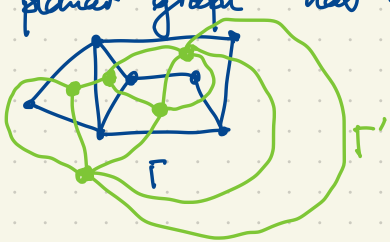


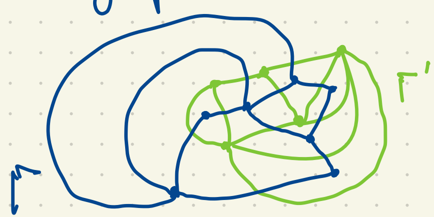
$K_{3,3}$

$K_{2,3}$ has no Hamilton circuit so we can't use the circle-chord method; but it is planar.

Every planar graph has a dual graph which is also planar.

eg.

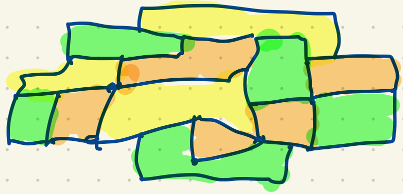


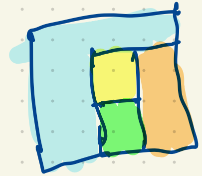


Γ' has $n' = 5$ vertices, $e' = 10$ edges, $r' = 7$ regions

This example Γ has n: 7 vertices, $e = 10$ edges, $r = 5$ regions. By Euler's formula (we'll prove soon), $n - e + r = 7 - 10 + 5 = 2$.

In drawing maps, the question arose: can we color political regions with 4 colors so that no two adjacent regions share the same color?





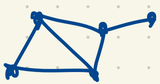
Theorem (Heawood) Every planar map can be properly colored using at most 5 colors.

Sigmund Freud

Theorem (Euler's Formula) Let Γ be a finite connected planar graph with n vertices, $e \geq 1$ edges, and r regions. (We will typically assume there are no loops or multiple edges although this is not strictly required.) Then $n - e + r = 2$. (This says the sphere has Euler characteristic 2, a theorem in topology.) In other words, $r = e - n + 2$.

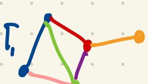
Proof by induction. Let Γ be constructed by a sequence of steps starting with a single edge $\Gamma_1 = \bullet\!\!-\!\!\bullet$; and by adding one edge at a time we reach $\Gamma = \Gamma_e$ when all e edges have been added. $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset \cdots \subset \Gamma_e = \Gamma$ are all connected.

eg.



Γ

Γ_1



Γ_2

Γ_3

Γ_4

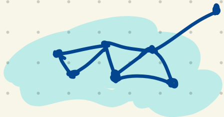
Γ_5

$\Gamma_6 = \Gamma$

Let's say Γ_i has n_i vertices, e_i edges, r_i regions.

Γ_1 $\bullet\!\!-\!\!\bullet$ has $n_1 = 2$ vertices, $e_1 = 1$ edge, $r_1 = 1$ region, and $n_1 - e_1 + r_1 = 2 - 1 + 1 = 2$. So Euler's formula holds for Γ_1.

Extending Γ_i to Γ_{i+1}, possibly we are adding one new vertex connected to Γ_i by the new edge:

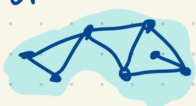


Γ_i

$$\left.\right\} \Gamma_{i+1} \qquad \begin{aligned} n_{i+1} &= n_i + 1 \\ e_{i+1} &= e_i + 1 \\ r_{i+1} &= r_i \end{aligned}$$

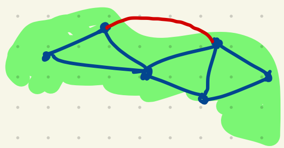
So $n_{i+1} - e_{i+1} + r_i = (n_i+1) - (e_i+1) + r_i = n_i - e_i + r_i = 2$

or



Another to case to consider is where Γ_{i+1} has the same vertices as Γ_i, but we are adding a new edge between two vertices that are already there.

Γ_i

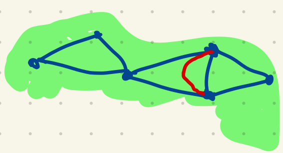


$$\left.\right\} \Gamma_{i+1} \qquad \begin{aligned} n_{i+1} &= n_i \\ e_{i+1} &= e_i + 1 \\ r_{i+1} &= r_i + 1 \end{aligned}$$

For Γ_{i+1}, $n_{i+1} - e_{i+1} + r_{i+1} = n_i - (e_i+1) + (r_i+1)$
$= n_i - e_i + r_i = 2$.

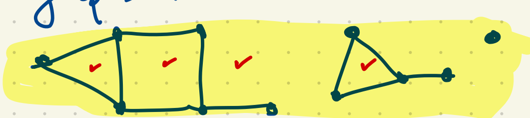
or

Γ_i



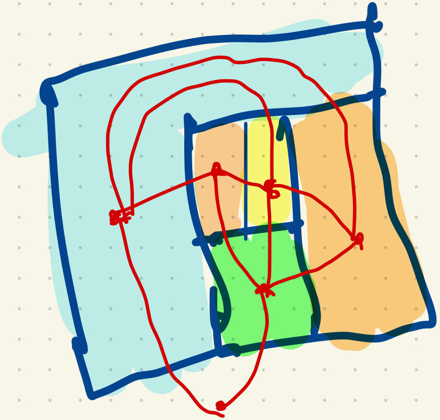
By induction, Euler's Formula holds for all $i = 1, 2, \dots, e$; in particular, the formula holds for $\Gamma_e = \Gamma$. $\square$

For graphs that are not connected, $n - e + r = 1 +$ number of connected components.



$n - e + r = 11 - 11 + 4 = 4 = 1 + 3$

$\uparrow$ number of connected components.



Every map in the plane corresponds to a planar graph.
Regions of the map give vertices of our graph.

The question is rephrased as follows:

Given a planar graph, what is the minimum number of colors required to properly color the vertices of the graph?
(A <u>proper</u> coloring of the vertices of a graph is one in which no edge has two endpoints of the same color.)