Combinatorics

Book 3

Fourth method: Decompose $F(x) = \frac{1+x}{1-x-x^2}$ using partial fractions.

Factor the denominator $1-x-x^2 = (1-\alpha x)(1-\beta x)$

Note: The factors $1-\alpha x$, $1-\beta x$ reveal the reciprocal roots α, β. (The roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.)

The roots are the same as the roots of x^2+x-1

i.e. $\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

The reciprocal roots are $\frac{2}{-1\pm\sqrt{5}} \cdot \frac{-1\mp\sqrt{5}}{-1\mp\sqrt{5}} = \frac{2(-1\mp\sqrt{5})}{1-5} = \frac{1\mp\sqrt{5}}{2}$

$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio)

$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

$\alpha+\beta = 1$

$\alpha-\beta = \sqrt{5}$

$\alpha\beta = -1$

α, β are reciprocal roots of x^2+x-1

$\frac{1}{\alpha^2} + \frac{1}{\alpha} - 1 = 0$

$1+\alpha-\alpha^2 = 0$

$\alpha^2 = \alpha+1$

$\beta^2 = \beta+1$

Always use α, β in the algebraic simplification.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{n=0}^{\infty}(\alpha x)^n + B\sum_{n=0}^{\infty}(\beta x)^n$$

$$= \sum_{n=0}^{\infty} \boxed{\underbrace{(A\alpha^n + B\beta^n)}_{a_n}} x^n$$

$a_n \sim A\alpha^n$ (exponential growth rate)

Use partial fractions to find A, B such that

$$\frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$\alpha^2 = \alpha + 1$
$\beta^2 = \beta + 1$

$1+x = A(1-\beta x) + B(1-\alpha x)$ Evaluate at $x = \frac{1}{\alpha}$, then at $\frac{1}{\beta}$.

$1 + \frac{1}{\alpha} = A\left(1 - \frac{\beta}{\alpha}\right)$ $\qquad 1 + \frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right)$

$\alpha \longleftrightarrow \beta$ interchanged by algebraic conjugation

$\alpha^2 = \alpha + 1 = A(\alpha - \beta) = \sqrt{5}A \Rightarrow A = \frac{\alpha^2}{\sqrt{5}}$ $\qquad B = -\frac{\beta^2}{\sqrt{5}}$

$\sqrt{5} \longleftrightarrow -\sqrt{5}$

$$a_n = A\alpha^n + B\beta^n = \frac{\alpha^2}{\sqrt{5}}\alpha^n - \frac{\beta^2}{\sqrt{5}}\beta^n = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}$$

As $n \to \infty$, $\beta^n \to 0$ since $|\beta| < 1$ so $a_n \sim A\alpha^n$ where $A = \frac{\alpha^2}{\sqrt{5}}$.

<u>Asymptotics</u> If $f(n), g(n) \to \infty$ as $n \to \infty$, we write $f(n) \sim g(n)$ (f is <u>asymptotic</u> to g) if $\frac{f(n)}{g(n)} \to 1$. This is different from "$\approx$". (approximately equal).

e.g. $\sqrt{n^2+10n} \to \infty$ as $n \to \infty$; $\sqrt{n^2+10n} \sim n$ as $n \to \infty$.

Check: $\frac{\sqrt{n^2+10n}}{n} = \sqrt{1+\frac{10}{n}} \to 1.$ $\qquad \left(\lim_{n\to\infty} \sqrt{1+\frac{10}{n}} = 1\right).$

$$\sqrt{n^2+10n} - n = \left(\sqrt{n^2+10n} - n\right) \cdot \frac{\sqrt{n^2+10n} + n}{\sqrt{n^2+10n} + n} = \frac{10n}{\sqrt{n^2+10n} + 10n} = \frac{10}{\sqrt{1+\frac{10}{n}} + 1} \to 5 \quad \text{as } n \to \infty.$$

$n^3+7n^2 \sim n^3$ as $n\to\infty$ Since $\frac{n^3+7n^2}{n^3} = 1+\frac{7}{n} \to 1$ as $n\to\infty$

yet $(n^3+7n^2)-n^3 = 7n^2 \to \infty$ as $n\to\infty$

In our case the convergence is stronger: not only is $a_n \sim A\alpha^n$ but moreover $a_n - A\alpha^n \to 0$. We can actually evaluate a_n by taking the closest integer to $A\alpha^n$.

$\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$

Another example of partial fraction decomposition:

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{1+2x-3x^2}{(1+x)(1+2ix)(1-2ix)} = \frac{A}{1+x}+\frac{B}{1+2ix}+\frac{C}{1-2ix}$$

$$= A\sum_{n=0}^{\infty}(-1)^n x^n + B\sum_{n=0}^{\infty}(-2i)^n x^n + C\sum_{n=0}^{\infty}(2i)^n x^n = \sum_{n=0}^{\infty}\Big[\underbrace{A(-1)^n+B(-2i)^n+C(2i)^n}_{a_n}\Big]x^n$$

OR $$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x}+\frac{Bx+C}{1+4x^2}$$

since $\frac{1}{1+x}, \frac{x}{1+4x^2}, \frac{1}{1+4x^2}$

How fast does a_n grow? The reciprocal roots of $1+x+4x^2+4x^3$ are $-1, -2i, 2i$. $|-1|=1$, $|\pm 2i| = 2$.

$a_n \sim c\,2^n$. From Maple it seems $a_n \sim \frac{1}{10}2^n$. No! look again:

$$a_n \sim \begin{cases} \frac{9}{5}\cdot 2^n, & \text{if } n\equiv 0 \bmod 4; \\ \frac{1}{10}2^n, & \text{if } n\equiv 1 \bmod 4; \\ -\frac{9}{5}2^n, & \text{if } n\equiv 2 \bmod 4; \\ -\frac{1}{10}\cdot 2^n, & \text{if } n\equiv 3 \bmod 4. \end{cases}$$

$$F(x)=\frac{1+2x-3x^2}{(1+x)(1+4x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+4x^2}=\frac{-\frac{4}{5}}{1+x}+\frac{\frac{1}{5}x+\frac{9}{5}}{1+4x^2}=-\frac{4}{5}\left(1-x+x^2-x^3+x^4-x^5+\cdots\right)$$
$$+\left(\frac{1}{5}x+\frac{9}{5}\right)\left(1-4x^2+16x^4-64x^6+\cdots\right)$$

Solve for A,B,C

$$\frac{1}{1-u}=1+u+u^2+u^3+u^4+\cdots$$

$$=\sum_{n=0}^{\infty}a_nx^n$$

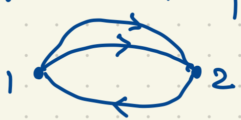
where $a_n=(-1)^{n+1}\frac{4}{5}+\begin{cases}\frac{9}{5}(-4)^{\frac{n}{2}} & \text{if } n \text{ is even}\\ \frac{1}{5}(-4)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.}\end{cases}$

Alternatively, (different constants A,B,C)

$$F(x)=\frac{A}{1+x}+\frac{B}{1+2ix}+\frac{C}{1-2ix}=\frac{-\frac{4}{5}}{1+x}+\frac{\frac{9}{5}-\frac{1}{10}i}{1+2ix}+\frac{\frac{9}{5}+\frac{1}{10}i}{1-2ix}$$ (Something like this... look at MAPLE session)

$|a_n|$ "grows exponentially" (const $\cdot\, 2^n$) but $a_n \not\sim c2^n$. This happens because the denominator of $F(x)$ has two reciprocal roots of the same largest absolute value.

Another example in counting walks in a graph where this issue arises:



$A:\begin{bmatrix}0&2\\1&0\end{bmatrix}$

$w_n=w_n(1,1)=$ number of walks of length n from vertex 1 to itself.

n	0	1	2	3	4	5	6	...
w_n	1	0	2	0	4	0	8	...

$$W(x)=\left[I-xA\right]^{-1}=\left[\begin{bmatrix}1&0\\0&1\end{bmatrix}-x\begin{bmatrix}0&2\\1&0\end{bmatrix}\right]^{-1}=\begin{bmatrix}1&-2x\\-x&1\end{bmatrix}^{-1}$$
$$=\frac{1}{1-4x^2}\begin{bmatrix}1&2x\\x&1\end{bmatrix}=\begin{bmatrix}W_{11}(x)&W_{12}(x)\\W_{21}(x)&W_{22}(x)\end{bmatrix}$$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

$$w(x) = w_{11}(x) = \frac{1}{1-4x^2} = 1+4x^2+16x^4+64x^6+256x^8+\cdots$$

$$w_n = w_n(1,1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even.} \end{cases}$$

Denominator $1-4x^2 = (1+2x)(1-2x)$ has two $\left\{\begin{array}{l}\text{reciprocal} \\ \text{roots}\end{array}\right.$ ± 2 having the same absolute value 2.

Remarks: $\frac{1}{1-4x^2}$ is preferred over $\frac{-\frac{1}{4}}{x^2-\frac{1}{4}}$ since we want to use the geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$

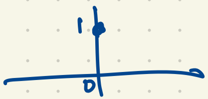
Exponential growth $f(n) \sim c\,a^n$ (c, a, k constants)

Polynomial growth $f(n) \sim c\,n^k$

eg. $4n^3+7n^2+11n+59 \sim 4n^3$

Other counting problems leading to a sequence where generating functions are used to express the solution:

Let a_n be the number of permutations of $[n] = \{1, 2, \ldots, n\}$ (i.e. the number of ways I can list n students in order). Then $a_n = n!$. Its generating function is



$$F(x) = \sum_{n=0}^{\infty} n!\,x^n = 1+x+2x^2+6x^3+24x^4+120x^5+720x^6+5040x^7+\cdots$$

$$G(x) = \sum_{n=0}^{\infty} (n!)^2 x^n = 1+x+4x^2+36x^3+576x^4+\cdots$$

$\binom{n}{k}$ is the number of k-subsets of an n-set

i.e. the number of bitstrings of length n having k 1's (and $n-k$ zeroes).

If $a_k = \binom{n}{k}$ where n is fixed then the generating function for the sequence $a_0, a_1, a_2, \ldots$ is

$$A_n(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

eg. $A_4(x) = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \cdots = 1 + 4x + 6x^2 + 4x^3 + x^4 = (1+x)^4$ Binomial Theorem

The Binomial Theorem $(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$ holds for all real values of m.

If m is a non-negative integer then $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is a non-negative integer (positive for $n = 0, 1, 2, \ldots, m$; zero for $n > m$) in which case $(1+x)^m$ is a polynomial in x of degree m. This is a special case of the Binomial Series.

The Binomial coefficients are found by hand from Pascal's Triangle

$$\begin{array}{ccccccccccccccc}
 & & & & & & 0 & & 1 & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
 & & & & & & & \cdots & & & & & & &
\end{array}$$

$\binom{m}{n}$ = entry n in row m of Pascal's Triangle

eg. $\binom{4}{2}$ = entry 2 in row 4

(start counting at $0, 1, 2, \ldots$)

The recursive formula for generating Pascal's Triangle is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\begin{tikzcd} \binom{n-1}{k-1} \arrow[dr] & & \binom{n-1}{k} \arrow[dl] \\ & \binom{n}{k} & \end{tikzcd}$$

Three proofs of Pascal's formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$:

<u>Combinatorial Proof</u> (counting proof): Consider the n-set $[n] = \{1, 2, \dots, n\}$.

Any k-subset $B \subseteq [n]$ is of one of the following two types:

(i) $n \in B$. In this case $B = \{n\} \cup B'$ where $B' \subseteq [n-1]$, $|B'| = k-1$.

There are $\binom{n-1}{k-1}$ ways to choose B' in this case.

(ii) $n \notin B$. In this case $B \subseteq [n-1]$. There are $\binom{n-1}{k}$ choices for B in this case.

The sum in cases (i) and (ii) must give $\binom{n}{k}$. $\square$

<u>Generating Function Proof</u>: Compare coefficients of x^k on both sides of

$$(1+x)^n = (1+x)(1+x)^{n-1}$$

$$1 + nx + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + x^n = (1 + x)\left(1 + (n-1)x + \dots \binom{n-1}{k-1}x^{k-1} + \binom{n-1}{k}x^k + \dots + x^{n-1}\right)$$

which gives $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. $\square$

Third Proof

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$n! = n \cdot (n-1)!$

$$= \frac{(n-1)!\,k}{(k-1)!\,(n-k)\,(n-k-1)!\,k} + \frac{(n-1)!\,(n-k)}{k\cdot(k-1)!\,(n-k-1)!\,(n-k)}$$

$$= \frac{(n-1)!\,k + (n-1)!\,(n-k)}{(k-1)!\,(n-k-1)!\,k\,(n-k)}$$

$$= \frac{n\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \qquad \square$$

$$A_n(x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

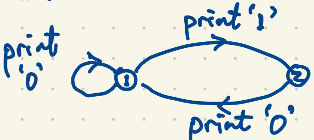
$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ = the sum of the entries in row n of Pascal's triangle.

A combinatorial explanation for this result is

2^n = number of subsets of $[n]$ = $\sum_{i=0}^{n}$ (number of i-subsets of $[n]$) = $\sum_{i=0}^{n} \binom{n}{i}$

(or 2^n = number of bitstrings of length n which can be rewritten as $\sum_{i=0}^{n} \binom{n}{i}$ where $\binom{n}{i}$ is the number of bitstrings of length n having exactly i 1's.)

HW3 #2 is similar to the example on the handout on Fibonacci numbers



$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

This directed graph is an example of a nondeterministic finite automaton with two states 1, 2.

How many walks are there starting at vertex 1? $w_n = w_n(1,1) + w_n(1,2)$

Printout 0101000 represents the walk $(1,2,1,2,1,1,1,1)$ of length 7

The walks of length n starting at vertex 1 are in one-to-one correspondence with 11-free bitstrings of length n.

More generally, many counting problems (where recursion plays a role) are equivalent to counting walks in graphs.

Recall: Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ where $\binom{n}{k}$ (binomial coefficient "n choose k") equals the number of k-subsets of an n-set. $\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } k \in \{0,1,2,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$ ($n \geq 0$ integer)

Multinomial Theorem $(x_1 + x_2 + \dots + x_r)^n = \sum_{i_1,\dots,i_r} \binom{n}{i_1, i_2, \dots, i_r} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$

$\binom{n}{i_1, i_2, \dots, i_r} = \frac{n!}{i_1! \, i_2! \cdots i_r!}$ if $i_1, \dots, i_r \geq 0$, $i_1 + \dots + i_r = n$; 0 otherwise

Multinomial Coefficient

eg. $(x+y+z)^3 = \sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} \binom{3}{i,j,k} x^i y^j z^k = x^3+y^3+z^3 + 3x^2y + 3xy^2 + 3x^2z + 3xz^2 + 3y^2z + 3yz^2 + 6xyz$

(Trinomial expansion)

$\binom{3}{3,0,0} = \frac{3!}{3!\,0!\,0!} = \frac{6}{6\cdot 1\cdot 1} = 1 = \binom{3}{0,3,0} = \binom{3}{0,0,3}$

$\binom{3}{2,1,0} = \frac{6}{2\cdot 1\cdot 1} = 3 = \binom{3}{0,2,1} = \cdots$

$\binom{3}{1,1,1} = \frac{3!}{1!\,1!\,1!} = \frac{6}{1\cdot 1\cdot 1} = 6$

Check: $3^3 = \underbrace{1+1+1}_{3} + \underbrace{3+\cdots+3}_{6} + 6 = 27$

(evaluating at $(1,1,1)$).

How many words can be formed by permuting the letters of MISSISSIPPI ?
(Words are strings of letters where the order is important.)

$$\frac{11!}{1!\,4!\,4!\,2!} = \binom{11}{1,4,4,2} = 34{,}650.$$

How many words can be formed by permuting the bits in 01110010010 ?

$$\binom{11}{5,6} = \frac{11!}{5!\,6!} = \binom{11}{5} = \binom{11}{6} = 462$$

Say M&M's are made in 6 different colors. How many different ways can we have a handful of 10 M&Ms? or n M&M's ?

a_n = number of ways to have a handful of n M&Ms ?

n	0	1	2	$\cdots$
a_n	1	6	21	$\cdots$

If M&M's come in the colors red ("R"), blue ("Bl"), green ("G"), orange ("O"), yellow ("Y"), brown ("Br"), then there are $\binom{15}{5}$ ways to draw a handful of ten M&M's e.g.

"X" is a divider

R R X X G X O O O O X Y X Br Br

$\underbrace{**}_{\text{red}}\ \overset{\uparrow}{\underset{\text{blue}}{}}XX\ \underbrace{*}_{\text{green}}\ X\ \underbrace{****}_{\text{orange}}\ X\ \underbrace{*}_{\text{yellow}}\ X\ \underbrace{**}_{\text{brown}}$ represents the color distribution

2 red
0 blue
1 green
4 orange
1 yellow
2 brown

10 M&M's

The possible color distributions for a handful of 10 M&M's are in one-to-one correspondence with the number of words of length 15 over a binary alphabet '*', 'X'. So the number of handfuls of 10 M&M's which come in 6 colors is $\binom{15}{5}$.

If M&M's come in k colors and we select n M&M's from this batch, the number of possible color distributions is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

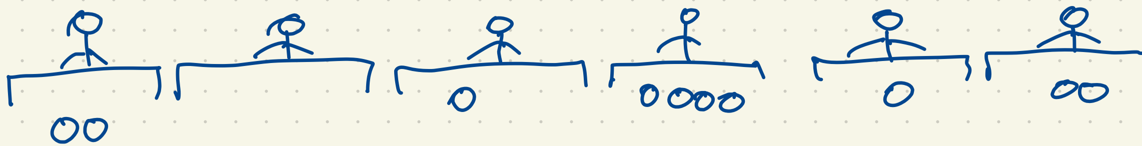
① Suppose I want to hand out n books (all different) to k students. How many ways can I do this?

$$\underbrace{k \times k \times \cdots \times k}_{n \text{ times}} = k^n \text{ choices.}$$

$\binom{n+k-1}{n}$

② How many ways can I hand out n identical silver dollars to k students?

Eg. I hand out 10 identical silver dollars to 6 students.



$\leftrightarrow$ $\circ\circ \,|\,|\, \circ \,|\, \circ\circ\circ\circ \,|\, \circ \,|\, \circ\circ$

Answer: $\binom{15}{5} = \binom{15}{10}$

Note: Problem ① is counting functions $[n] \longrightarrow [k]$.

In Problem ②, what if we require each student to get at least one of the silver dollars? Instead of $\binom{10+6-1}{10}$, the answer is $\binom{4+6-1}{4} = \binom{9}{4}$.

Suppose I want to hand out k different books to n students, in such a way that each student gets at most one book. How many ways can we distribute the books?

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

This equals zero if $k > n$.

no. of choices of student to give book 1 to — 2nd book — 3rd — k^{th} book

$P(n,k) = 0$ if $k < n$

$P(n,k) = n!$ if $k = n$

$P(n,k)$ is also denoted $n_{(k)}$ or various other notations

("descending factorial" or "falling faction")

$P(n,k)$ is the number of one-to-one maps $[k] \longrightarrow [n]$ (injections)

Question: How many surjections $[k] \longrightarrow [n]$? (functions that are onto, i.e. how many ways can we hand out k different books to n students if we want every student to get at least one book)?

Binomial Theorem $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

What if m is not an integer?

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!} = \frac{m\cdot(m-1)(m-2)\cdots(m-k+1)\cancel{(m-k)(m-k-1)\cdots 1}}{k!\ \cancel{(m-k)(m-k-1)(m-k-2)\cdots 1}} = \frac{P(m,k)}{k!}$$

$P(m,k) = m(m-1)(m-2)\cdots(m-k+1)$ is defined for all $k \in \{0,1,2,3,4,\dots\}$ and m any real number.

$P(m,0) = 1$

$P(m,1) = m$

$P(m,2) = m(m-1)$