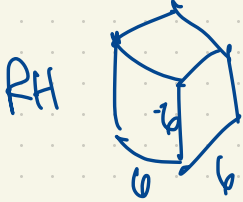




# Combinatorics

Book 1

Eric

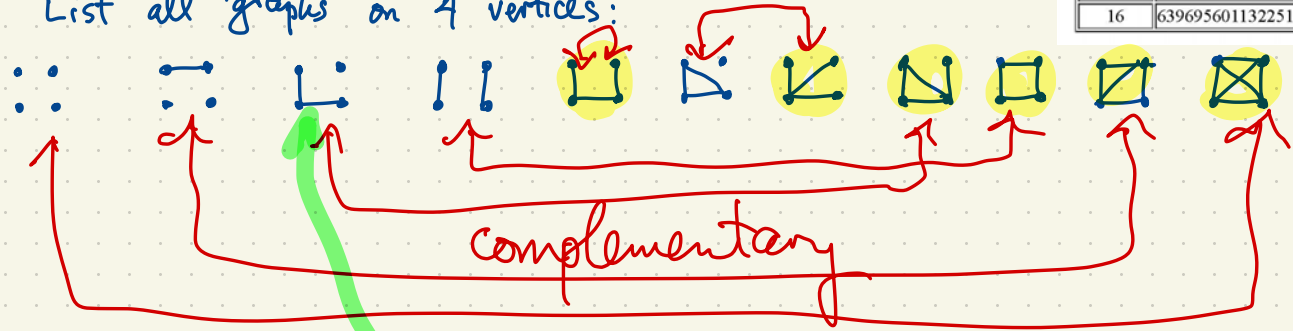


$$6^3 = 216$$

#vertices	Connected graphs	All graphs
1	1	1
2	1	2
3	2	4
4	6	11
5	21	34
6	112	156
7	853	1044
8	11117	12346
9	261080	274668
10	11716571	12005168
11	1006700565	1018997864
12	164059830476	165091172592
13	50335907869219	50502031367952
14	29003487462848061	29054155657235488
15	31397381142761241960	31426485969804308768
16	63969560113225176176277	64001015704527557894928

Ordinary / Simple Graph on n vertices/nodes

Eg. List all  $\cong$  graphs on 4 vertices:

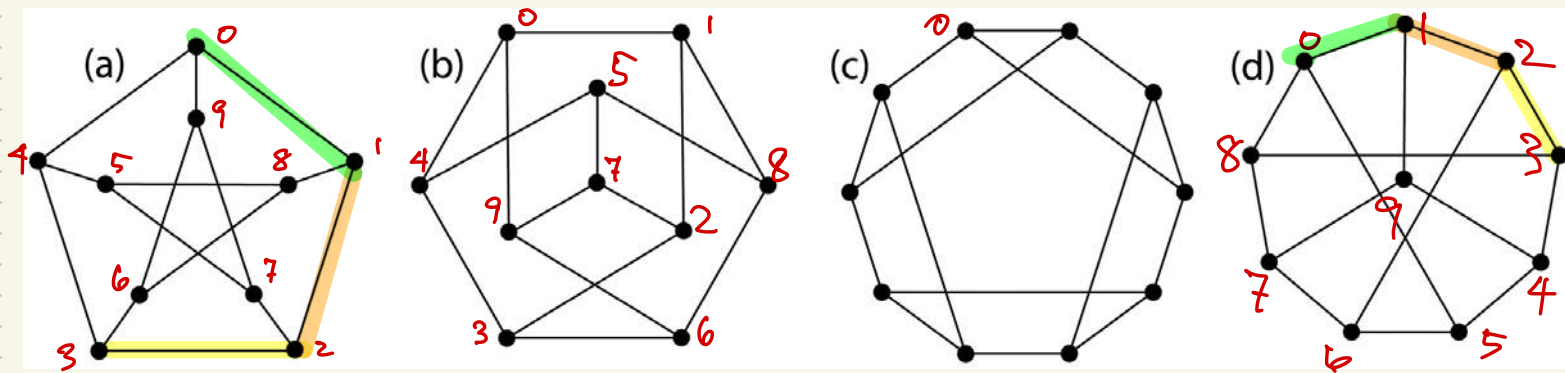


A graph of order  $n$  is a pair  $G = (V, E)$  where  $V$  is a set of  $n$  vertices and  $E$  is a subset of pairs  $\{v, w\}$  where  $v \neq w, v, w \in V$ .  
 and edges  $\{1, 3\}, \{2, 3\}$  can be illustrated

Eg. the graph with vertices  $1, 2, 3, 4$

(the two graphs are isomorphic).





Of these four graphs, which one is not isomorphic to the others?  
 Graphs (a), (b) are isomorphic. Graph (c) is not isomorphic to (a) or (b) because graph (a) has diameter 2: any two vertices are at distance at most 2 apart. However, graph (c) has diameter 3.


A symmetry automorphism of a graph is an isomorphism from the graph to itself.

An isomorphism from graph (a) to graph (d) is the map with table of values

vertex in (a)	vertex in (d)
0	0
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9

This is a very special graph having the special property that for every path of length 3 (vertices  $v_0, v_1, v_2, v_3$  with  $v_0 \sim v_1 \sim v_2 \sim v_3$ ,  $v_0 \not\sim v_2$ ,  $v_0 \not\sim v_3$ ,  $v_1 \not\sim v_3$ ) in (a) and every path  $w_0 \sim w_1 \sim w_2 \sim w_3$  in (d) ( $w_0 \not\sim w_2$ ,  $w_0 \not\sim w_3$ ,  $w_1 \not\sim w_3$ ) there is a unique isomorphism (a)  $\rightarrow$  (d) mapping  $v_i \mapsto w_i$ .  
 This is a Petersen graph. How many isomorphisms are there from (a) to (d)?  
 $10 \times 3 \times 2 \times 2 = 120$ .

In particular, a Petersen graph has 120 automorphisms.

The graph  (a 4-cycle) has 8 automorphisms

$0 \mapsto 1$	$0 \mapsto 0$	$0 \mapsto 0$
$1 \mapsto 2$	$1 \mapsto 3$	$1 \mapsto 1$
$2 \mapsto 3$	$2 \mapsto 2$	$2 \mapsto 2$
$3 \mapsto 0$	$3 \mapsto 1$	$3 \mapsto 3$

identity

Not an automorphism:

$0 \mapsto 0$
$1 \mapsto 1$
$2 \mapsto 3$
$3 \mapsto 2$

The edge  $0 \mapsto 3$  is mapped to a non-edge  $0 \mapsto 2$

The graph  has exactly 2 automorphisms

A graph with only one automorphism? • (the graph of order 1, i.e. having only one vertex).  
A less trivial example with more than one vertex:



Every graph has a degree sequence. The degree of a vertex is the number of its neighbors.

The graph  $\Gamma$  (above) has degree sequence  $(1, 1, 1, 2, 2, 3)$ .  $1+1+1+2+2+3=12$

If two graphs are isomorphic, they must have the same degree sequence.

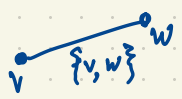
An isomorphism from  $\Gamma$  to  $\Gamma'$  must map each vertex to a vertex of the same degree.

If two graphs have the same degree sequence, must they be isomorphic? **No**, e.g. the graphs (a), (c) on the previous page are not isomorphic, but both have degree sequence  $(3, 3, 3, 3, 3, 3, 3, 3)$ .

A graph with  $n$  vertices and  $e$  edges has order  $n$ . The degree of vertex  $v$ , denoted  $\deg(v)$ , is the number of vertices joined to  $v$ . If  $G$  has vertices labelled  $1, 2, 3, \dots, n$ , then the degree sequence of  $G$  is  $(\deg(1), \deg(2), \dots, \deg(n))$ , permuted into increasing order. A graph  $G$  is  $d$ -regular if  $\deg(v) = d$  for every vertex  $v$  in  $G$  (or simply regular). Note:  $\deg(1) + \deg(2) + \dots + \deg(n) = 2e$ .

Theorem If  $G$  is a (finite) simple graph with  $e$  edges, then  $\sum_{v \in V} \deg(v) = 2e$  where  $G = (V, E)$ ,  
 $V$  the set of vertices,  $E$  the set of edges.

Proof We count in two different ways the number of pairs  $(v, \{v, w\})$  in  $G$  ( $v \in V, \{v, w\} \in E$ ).



Since every edge  $\{v, w\}$  has two vertices  $v, w$ , there are  $2e$  such pairs.

On the other hand, since each vertex  $v \in V$  has  $\deg(v)$  edges, we have

$\sum_{v \in V} \deg(v)$  as the number of such pairs. These answers must agree.  $\square$

Imagine we organize a round robin <sup>fencing</sup> tournament between  $n$  competitors. Every competitor competes with each of the others exactly once. Altogether there are  $\binom{n}{2} = \frac{n(n-1)}{2}$ .

In general  $\binom{n}{k}$  = "n choose k" is the number of ways to choose a  $k$ -subset of an  $n$ -set (i.e. a subset of size  $k$  in a set of  $n$  elements).  $\binom{n}{k}$  is a binomial coefficient.

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad (\text{the Binomial Theorem})$$

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b) = aaaaa + qaqaab + qaabaq + qaaqbb + \dots + bbbbbb$$

Before collecting terms, there are  $2^5 = 32$  terms.

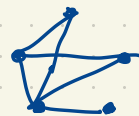
$$\text{Pascal's Triangle} = \binom{5}{0} a^5 b^0 + \binom{5}{1} a^4 b^1 + \binom{5}{2} a^3 b^2 + \binom{5}{3} a^2 b^3 + \binom{5}{4} a b^4 + \binom{5}{5} a^0 b^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$



Theorem In a simple graph with  $n \geq 2$  vertices, there exist two vertices of the same degree.

Proof Let  $(d_1, d_2, \dots, d_n)$  be the degree sequence of a graph of order  $n \geq 2$ . Note that  $d_1, \dots, d_n \in \{0, 1, 2, \dots, n-1\}$ . If  $d_1, \dots, d_n$  are distinct then every element of  $\{0, 1, 2, \dots, n-1\}$  is the degree of some vertex by the Pigeonhole Principle. This means the degree sequence is  $(0, 1, 2, \dots, n-1)$ . In particular, there is a vertex of degree 0, and a vertex of degree  $n-1$ , a contradiction. This proves the result.  $\square$



degrees 1, 2, 2, 3, 4

Note:  $i \mapsto d_i$   
 $\{1, 2, \dots, n\} \rightarrow \{0, 1, 2, \dots, n-1\}$ .

Proofs are logical arguments that argue the truth of our assertion. They are always written in proper sentences.

singular	plural
vertex	vertices
index	indices
matrix	matrices

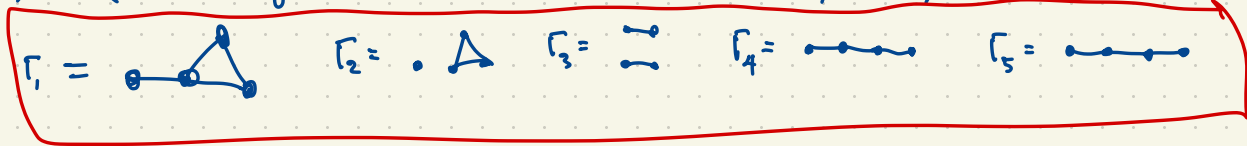
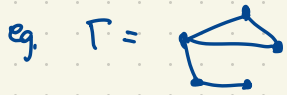
• !  
 has degree sequence  $(0, 1, 1)$ .  
 $\{0, 1\}$  is the set of degrees of the vertices

Pigeonhole Principle Suppose  $n$  pigeons come to roost in  $k$  holes. If  $n > k$ , then <sup>(at least)</sup> two pigeons must be in the same hole. If  $n \leq k$ , at least one of the holes will be empty. In other words, if  $f: A \rightarrow B$  is any function where  $|A| = n$  and  $|B| = k$ , then: (i) if  $n > k$  then  $f$  cannot be one-to-one; (ii) if  $n < k$  then  $f$  cannot be onto. (iii) Assuming  $n = k$  then  $f$  is one-to-one iff it is onto.

## Graph Reconstruction Problem

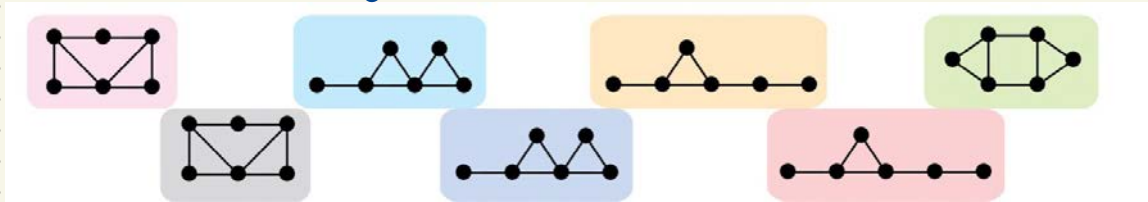
Starting with a (simple) graph  $\Gamma$  of order  $n$ , we construct a set of  $n$  graphs  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  where  $\Gamma_i$  is formed by deleting vertex  $i$  (and all edges from vertex  $i$ ). The set  $\{\Gamma_1, \Gamma_2, \dots, \Gamma_n\}$  is called the deck of  $\Gamma$ .

actually multiset



Can you (uniquely) reconstruct  $\Gamma$  from its deck?

Consider this set of seven graphs of order 6. Find a graph  $\Gamma$  of order 7 having this as its deck.



Note: From the deck of any graph  $\Gamma$ , we can reconstruct (deduce) the degree sequence of  $\Gamma$ .

Given two graphs of order  $n$ , how hard is it to check whether they are isomorphic?

Assuming  $\Gamma, \Gamma'$  are given, each with  $n$  vertices, label the vertices of each graph  $1, 2, 3, \dots, n$ . The number of bijections from the vertices of  $\Gamma$  to the vertices of  $\Gamma'$  is  $n! = 1 \times 2 \times 3 \times \dots \times n$  ( $n$  factorial). (eg.  $1! = 1, 2! = 2, 3! = 6, 4! = 24, \dots, 10! = 3628800, \dots$ ). Check each of the bijections to see if it is an isomorphism. This takes at most  $n! \binom{n}{2}$ .



We have an algorithm for testing graph isomorphism but it requires (in the worst case)  $n! \binom{n}{2}$  steps where  $n$  is the order of the graphs.

$n! \rightarrow \infty$  faster than any polynomial in  $n$  i.e. if  $f(n)$  is a polynomial in  $n$  (eg.  $\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$ )

where  $k$  is constant... ( $\binom{n}{k}$  is a polynomial of degree  $k$  in  $n$ .)

i.e.  $\lim_{n \rightarrow \infty} \frac{n!}{f(n)} = \infty$  for any positive polynomial function  $f(n)$ .

In fact,  $n! \rightarrow \infty$  faster than any exponential function  $c^n$  ( $c > 1$ ) eg.

$$\lim_{n \rightarrow \infty} \frac{n!}{10^n} = \lim_{n \rightarrow \infty} \left( \frac{1}{10} \cdot \frac{2}{10} \cdot \frac{3}{10} \dots \frac{9}{10} \cdot \frac{10}{10} \cdot \frac{11}{10} \cdot \frac{12}{10} \cdot \frac{13}{10} \dots \frac{n}{10} \right) = \infty$$

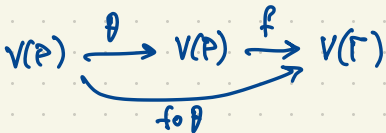
The best algorithms known for testing for graph isomorphism require far fewer than  $n! \binom{n}{2}$  steps (even in the worst case). These algorithms have running time that is intermediate between polynomial and exponential.

In the worst case, it takes  $O(n^2)$  steps to compute the degree sequence of a graph, a polynomial function of  $n$ .

Assume graph  $P$  (the Petersen graph) has 120 automorphisms. ( $P \cong \text{graph}(a)$ )

If  $\Gamma$  is any graph, then either  $P \not\cong \Gamma$  or there are <sup>exactly</sup> 120 isomorphisms  $P \rightarrow \Gamma$ .

If  $f: \underbrace{V(P)}_{\text{vertices of } P} \rightarrow \underbrace{V(\Gamma)}_{\text{vertices of } \Gamma}$  is an isomorphism then for every automorphism  $\theta: V(P) \rightarrow V(P)$ , we have an isomorphism



Given two graphs  $\Gamma, \Gamma'$ , there may be no isomorphism from  $\Gamma$  to  $\Gamma'$ . But if there is an isomorphism  $f: \Gamma \rightarrow \Gamma'$ , then the number of isomorphisms  $\Gamma \rightarrow \Gamma'$  is equal to the number of automorphisms of  $\Gamma$ :

$$\text{Aut}(\Gamma) = \{ \text{automorphisms of } \Gamma \} \xleftrightarrow{1:1} \{ \text{isomorphisms } \Gamma \rightarrow \Gamma' \}.$$

Given  $\theta \in \text{Aut } \Gamma$ ,

$$\Gamma \xrightarrow{\theta} \Gamma \xrightarrow{f} \Gamma' \quad \text{fo } \theta: \Gamma \rightarrow \Gamma' \text{ is an isomorphism.}$$

$\underbrace{\hspace{10em}}_{\text{fo } \theta}$

$\text{Aut } \Gamma$  is a group:

$$\begin{aligned} \iota: \Gamma &\rightarrow \Gamma \quad \text{identity} \\ \iota(x) &= x \quad \text{for every} \\ &\quad \text{vertex } x \text{ of } \Gamma \\ \iota &\in \text{Aut } \Gamma \end{aligned}$$

$$\iota \circ \theta = \theta = \theta \circ \iota \quad \text{for all } \theta \in \text{Aut } \Gamma;$$

$$(p \circ \sigma) \circ \tau = p \circ (\sigma \circ \tau) \quad \text{for all } p, \sigma, \tau \in \text{Aut } \Gamma;$$

for every  $\theta \in \text{Aut } \Gamma$   
there exists  $\theta^{-1} \in \text{Aut } \Gamma$   
such that  $\theta \circ \theta^{-1} = \iota = \theta^{-1} \circ \theta$ .

The map  $\theta \mapsto \text{fo } \theta$  is one-to-one. Given any isomorphism  $g: \Gamma \rightarrow \Gamma'$ , there exists  $\theta \in \text{Aut } \Gamma$  such that  $g = \text{fo } \theta$ . Why?

$$\begin{aligned} \Gamma &\xrightarrow{g} \Gamma' \xrightarrow{f^{-1}} \Gamma \\ &\quad \underbrace{\hspace{10em}}_{\text{fo } g} \\ \text{fo } g: \Gamma &\rightarrow \Gamma \text{ is an isomorphism} \\ \text{i.e. } \theta &= f^{-1} \circ g \in \text{Aut } \Gamma \\ \text{and } \text{fo } \theta &= \text{fo}(f^{-1} \circ g) = \underbrace{(\text{fo } f^{-1})}_{\iota} \circ g = g. \end{aligned}$$

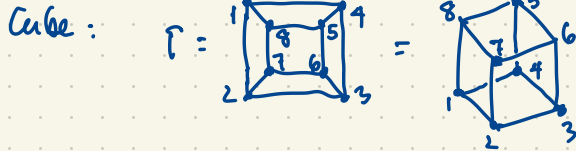
$n!$  grows faster than  $c^n$  for any  $c > 1$ ;  $n!$  grows slower than  $n^n$ :  
(exponential)

$$\lim_{n \rightarrow \infty} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \left( \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n} \right) = 0$$

Stirling's Approximation

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad \text{as } n \rightarrow \infty \quad \text{i.e. } \lim_{n \rightarrow \infty} \frac{n!}{\left(\frac{n}{e}\right)^n \sqrt{2\pi n}} = 1.$$

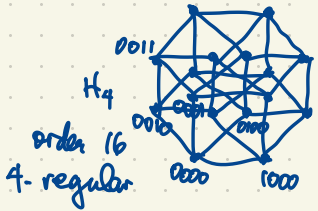
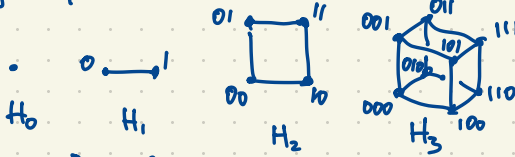
↑ asymptotic



How many automorphisms does  $\Gamma$  have?

The cube has 48 symmetries (24 rotational and 24 other)

Hamming Cube graph  $H_n$  has as its vertices all bitstrings of length  $n$ . A bit is a 0 or a 1 in the binary alphabet  $\{0,1\}$ . Two bitstrings are adjacent if they differ in exactly one position.  $H_n$  is a regular graph degree  $n$  with  $2^n$  vertices.



$H_n$  has diameter  $n$  and  $|\text{Aut } H_n| = 2^n n!$

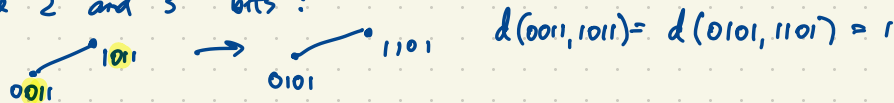
In  $H_4$ , switch 2nd and 4th coordinates.

$$d(0011, 1011) = d(0110, 1110) = 1$$

The  $n!$  permutations of the coordinates of the strings give  $n!$  automorphisms of  $H_n$ .

Also there are  $2^n$  possible bit flip operations (flip a single coordinate  $0 \leftrightarrow 1$ , or do this for a subset of the coordinates).

Eg. Flip the 2<sup>nd</sup> and 3<sup>rd</sup> bits:



Eg.  $|\text{Aut } H_3| = 2^3 3! = 8 \cdot 6 = 48$  as above.

Some infinite graphs: ...  ... 2-regular, connected  
 $H_{\infty}$  has as its vertices the bitstrings  $a_1 a_2 a_3 a_4 \dots$ ,  $a_i \in \{0, 1\}$  e.g.



$H_{\infty}$  is not connected.

Random infinite graphs are connected. They have diameter 2.

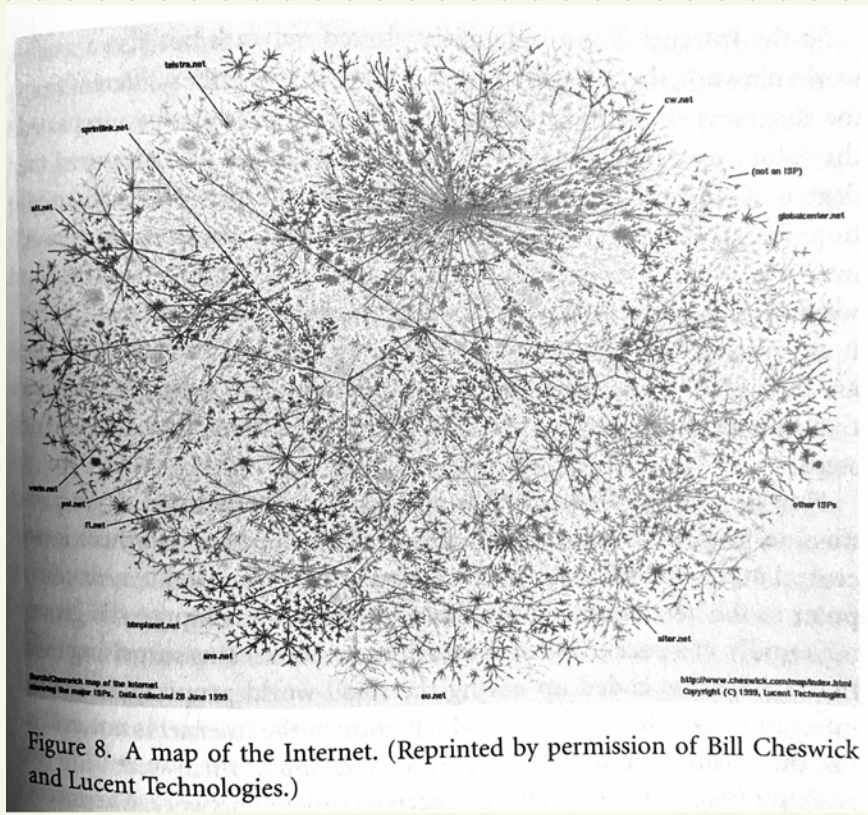
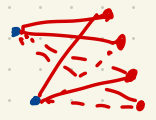
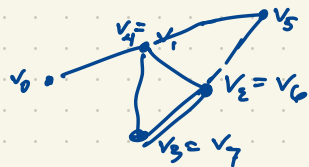


Figure 8. A map of the Internet. (Reprinted by permission of Bill Cheswick and Lucent Technologies.)

A walk in a graph  $\Gamma$  is a sequence of vertices  $(v_0, v_1, v_2, \dots, v_r)$  such that  $v_i$  is adjacent to  $v_{i+1}$  for  $i=0, 1, 2, \dots, r-1$ . This walk has length  $r$ .

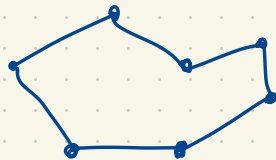


A path is a walk without repeating vertices.

A trail is a walk which possibly repeats vertices but does not repeat edges.

A circuit is a sequence of vertices  $(v_0, v_1, v_2, \dots, v_r)$  such that  $v_i \sim v_{i+1}$  for  $i=0, 1, 2, \dots, r-1$  where  $v_0, v_1, \dots, v_r$  are distinct except  $v_0 = v_r$ .

The complete graph  $K_n$  is the graph of order  $n$  in which  $x \sim y$  for every pair of distinct vertices  $x, y$ .



length 7.



Note:  $K_n$  has  $\binom{n}{2} = \frac{n(n-1)}{2}$  edges.

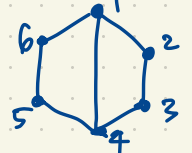
The complete bipartite graph  $K_{m,n}$  is the graph of order  $m+n$  and  $mn$  edges having vertex set  $V_0 \sqcup V_1$ .

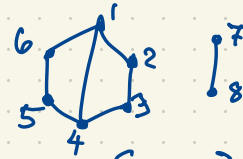
$(A \cup B$  is the union of two sets,  $A \cup B = \{x : x \in A \text{ or } x \in B\}$   
 $A \sqcup B$  is the disjoint union where  $A \cap B = \emptyset$  i.e.  $A, B$  are disjoint)



Given  $x, y \in V_0 \sqcup V_1$ , we have  $x \sim y$  iff one of  $x, y$  is in  $V_0$  and the other is in  $V_1$ .  
 Here  $|V_0| = m$ ,  $|V_1| = n$ .

A bipartite graph has vertex set  $V_0 \sqcup V_1$  and every edge has one endpoint in  $V_0$ , the other endpoint in  $V_1$ .

Eg.  is bipartite with partition  $\{1,3,5\} \sqcup \{2,4,6\}$ .



$\{1,3,5,7\} \sqcup \{2,4,6,8\}$

Theorem A graph is bipartite iff it has no circuits of odd length. or  $\{1,3,5,8\} \sqcup \{2,4,6,7\}$

A planar graph is a graph which can be drawn in the plane  $\mathbb{R}^2$  without crossing edges.

eg.  $K_4 =$    $=$   is planar.

eg.  $K_5 =$    $=$   is not planar.