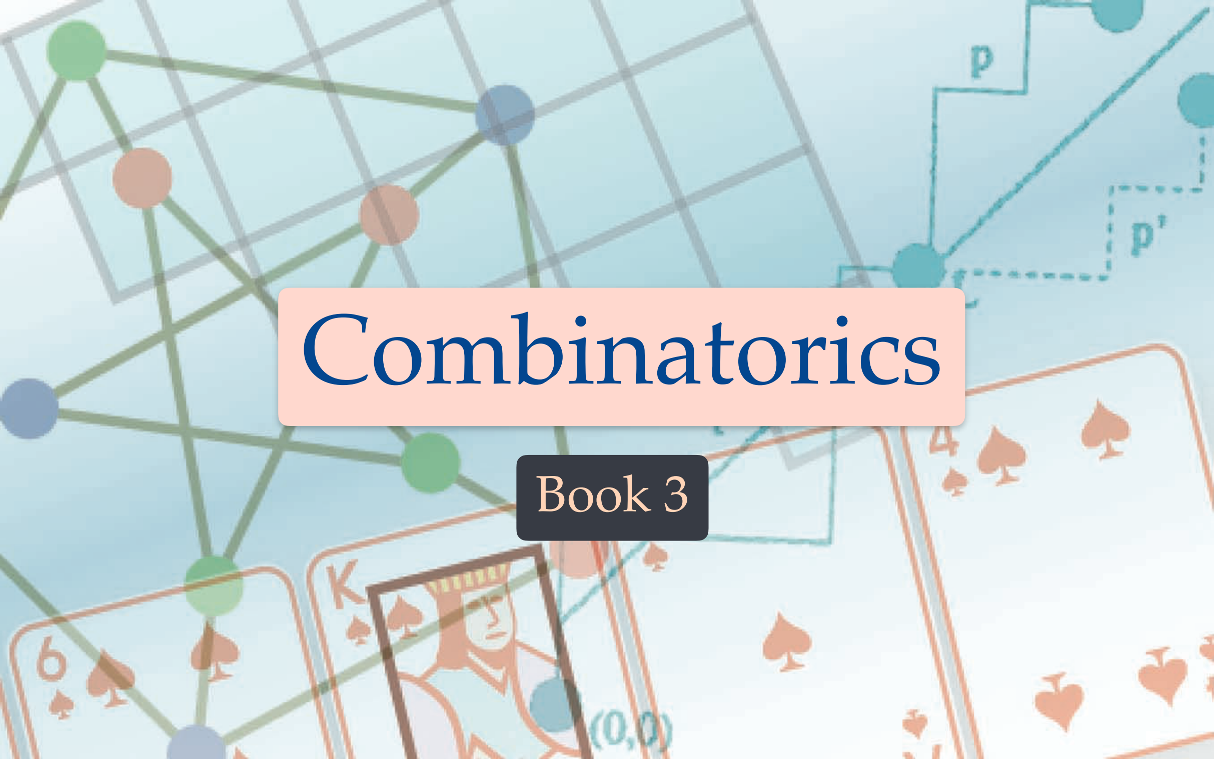


Combinatorics

Book 3

Fourth method: Decompose $F(x) = \frac{1+x}{1-x-x^2}$ using partial fractions.

Factor the denominator $1-x-x^2 = (1-\alpha x)(1-\beta x)$

Note: The factors $1-\alpha x$, $1-\beta x$ reveal the reciprocal roots α, β. (The roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.)

The roots are the same as the roots of x^2+x-1

i.e. $\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

The reciprocal roots are $\frac{2}{-1 \pm \sqrt{5}} \cdot \frac{-1 \mp \sqrt{5}}{-1 \mp \sqrt{5}} = \frac{2(-1 \mp \sqrt{5})}{1-5} = \frac{1 \mp \sqrt{5}}{2}$

$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio)

$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

$\alpha + \beta = 1$

$\alpha - \beta = \sqrt{5}$

$\alpha\beta = -1$

α, β are reciprocal roots of x^2+x-1

$\frac{1}{\alpha^2} + \frac{1}{\alpha} - 1 = 0$

$1 + \alpha - \alpha^2 = 0$

$\alpha^2 = \alpha + 1$

$\beta^2 = \beta + 1$

Always use α, β in the algebraic simplification.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A\sum_{n=0}^{\infty} (\alpha x)^n + B \sum_{n=0}^{\infty} (\beta x)^n$$

$$= \sum_{n=0}^{\infty} \boxed{\underbrace{(A\alpha^n + B\beta^n)}_{a_n}} x^n$$

$a_n \sim A\alpha^n$ (exponential growth rate)

Use partial fractions to find A, B such that

$$\frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$\alpha^2 = \alpha + 1$
$\beta^2 = \beta + 1$

$$1+x = A(1-\beta x) + B(1-\alpha x)$$

Evaluate at $x = \frac{1}{\alpha}$, then at $\frac{1}{\beta}$.

$$1 + \frac{1}{\alpha} = A\left(1 - \frac{\beta}{\alpha}\right) \qquad 1 + \frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right)$$

$$\alpha^2 = \alpha + 1 = A(\alpha - \beta) = \sqrt{5}A \implies A = \frac{\alpha^2}{\sqrt{5}} \qquad B = -\frac{\beta^2}{\sqrt{5}}$$

$\alpha \longleftrightarrow \beta$ interchanged by algebraic conjugation

$\sqrt{5} \longleftrightarrow -\sqrt{5}$

$$a_n = A\alpha^n + B\beta^n = \frac{\alpha^2}{\sqrt{5}}\alpha^n - \frac{\beta^2}{\sqrt{5}}\beta^n = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}$$

As $n \to \infty$, $\beta^n \to 0$ since $|\beta| < 1$ so $a_n \sim A\alpha^n$ where $A = \frac{\alpha^2}{\sqrt{5}}$.

<u>Asymptotics</u> If $f(n), g(n) \to \infty$ as $n \to \infty$, we write $f(n) \sim g(n)$ (f is <u>asymptotic</u> to g) if $\frac{f(n)}{g(n)} \to 1$. This is different from "$\approx$". (approximately equal).

eg. $\sqrt{n^2+10n} \to \infty$ as $n \to \infty$; $\sqrt{n^2+10n} \sim n$ as $n \to \infty$.

Check: $\frac{\sqrt{n^2+10n}}{n} = \sqrt{1+\frac{10}{n}} \to 1$. $\left(\lim_{n\to\infty} \sqrt{1+\frac{10}{n}} = 1\right)$.

$$\sqrt{n^2+10n} - n = \left(\sqrt{n^2+10n} - n\right) \cdot \frac{\sqrt{n^2+10n} + n}{\sqrt{n^2+10n} + n} = \frac{10n}{\sqrt{n^2+10n} + 10n} = \frac{10}{\sqrt{1+\frac{10}{n}} + 1} \to 5 \quad \text{as } n \to \infty.$$

$n^3+7n^2 \sim n^3$ as $n\to\infty$ Since $\frac{n^3+7n^2}{n^3} = 1+\frac{7}{n} \to 1$ as $n\to\infty$

yet $(n^3+7n^2)-n^3 = 7n^2 \to \infty$ as $n\to\infty$

In our case the convergence is stronger: not only is $a_n \sim A\alpha^n$ but moreover $a_n - A\alpha^n \to 0$. We can actually evaluate a_n by taking the closest integer to $A\alpha^n$.

$\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$

Another example of partial fraction decomposition:

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{1+2x-3x^2}{(1+x)(1+2ix)(1-2ix)} = \frac{A}{1+x}+\frac{B}{1+2ix}+\frac{C}{1-2ix}$$

$$= A\sum_{n=0}^{\infty}(-1)^n x^n + B\sum_{n=0}^{\infty}(-2i)^n x^n + C\sum_{n=0}^{\infty}(2i)^n x^n = \sum_{n=0}^{\infty}\Big[\underbrace{A(-1)^n+B(-2i)^n+C(2i)^n}_{a_n}\Big]x^n$$

OR

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x}+\frac{Bx+C}{1+4x^2}$$

since $\frac{1}{1+x}$, $\frac{x}{1+4x^2}$, $\frac{1}{1+4x^2}$

How fast does a_n grow? The reciprocal roots of $1+x+4x^2+4x^3$ are -1, $-2i$, $2i$. $|-1|=1$, $|\pm 2i| = 2$.

$a_n \sim c\,2^n$. From Maple it seems $a_n \sim \frac{1}{10}2^n$.

No! look again:

$$a_n \sim \begin{cases} \frac{9}{5}\cdot 2^n, & \text{if } n\equiv 0 \bmod 4;\\ \frac{1}{10}2^n, & \text{if } n\equiv 1 \bmod 4;\\ -\frac{9}{5}2^n, & \text{if } n\equiv 2 \bmod 4;\\ -\frac{1}{10}\cdot 2^n, & \text{if } n\equiv 3 \bmod 4.\end{cases}$$

$$F(x) = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+4x^2} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{1}{5}x + \frac{9}{5}}{1+4x^2} = -\frac{4}{5}\left(1-x+x^2-x^3+x^4-x^5+\cdots\right)$$
$$+ \left(\tfrac{1}{5}x + \tfrac{9}{5}\right)\left(1-4x^2+16x^4-64x^6+\cdots\right)$$

Solve for A, B, C

$$\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = (-1)^{n+1}\frac{4}{5} + \begin{cases} \frac{9}{5}(-4)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{1}{5}(-4)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$

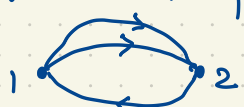
Alternatively, (different constants A, B, C)

$$F(x) = \frac{A}{1+x} + \frac{B}{1+2ix} + \frac{C}{1-2ix} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{9}{5} - \frac{1}{10}i}{1+2ix} + \frac{\frac{9}{5} + \frac{1}{10}i}{1-2ix}$$

(Something like this... look at MAPLE session)

$|a_n|$ "grows exponentially" (const $\cdot\, 2^n$) but $a_n \not\sim c2^n$. This happens because the denominator of $F(x)$ has <u>two</u> reciprocal roots <u>of the same</u> largest absolute value.

Another example in counting walks in a graph where this issue arises:



$A: \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$w_n = w_n(1,1) =$ number of walks of length n from vertex 1 to itself.

n	0	1	2	3	4	5	6	$\cdots$
w_n	1	0	2	0	4	0	8	$\cdots$

$$W(x) = [I - xA]^{-1} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - x\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1 & -2x \\ -x & 1 \end{bmatrix}^{-1}$$
$$= \frac{1}{1-4x^2}\begin{bmatrix} 1 & 2x \\ x & 1 \end{bmatrix} = \begin{bmatrix} W_{11}(x) & W_{12}(x) \\ W_{21}(x) & W_{22}(x) \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$w(x) = w_{11}(x) = \frac{1}{1-4x^2} = 1+4x^2+16x^4+64x^6+256x^8+\cdots$$

$$w_n = w_n(1,1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even.} \end{cases}$$

Denominator $1-4x^2 = (1+2x)(1-2x)$ has two $\left\{\begin{matrix}\text{reciprocal}\\ \text{roots}\end{matrix}\right.$ ± 2 having the same absolute value 2.

__Remarks__: $\frac{1}{1-4x^2}$ is preferred over $\frac{-\frac{1}{4}}{x^2-\frac{1}{4}}$ since we want to use the geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$

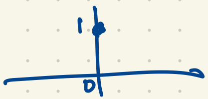
Exponential growth $f(n) \sim c a^n$ $\quad$ (c, a, k constants)

Polynomial growth $f(n) \sim c n^k$

eg. $4n^3+7n^2+11n+59 \sim 4n^3$

Other counting problems leading to a sequence where generating functions are used to express the solution:

Let a_n be the number of permutations of $[n] = \{1, 2, \ldots, n\}$ (i.e. the number of ways I can list n students in order). Then $a_n = n!$. Its generating function is



$$F(x) = \sum_{n=0}^{\infty} n!\, x^n = 1+x+2x^2+6x^3+24x^4+120x^5+720x^6+5040x^7+\cdots$$

$$G(x) = \sum_{n=0}^{\infty} (n!)^2 x^n = 1+x+4x^2+36x^3+576x^4+\cdots$$

$\binom{n}{k}$ is the number of k-subsets of an n-set
i.e. the number of bitstrings of length n having k 1's (and n-k zeroes).

If $a_k = \binom{n}{k}$ where n is fixed then the generating function for the sequence $a_0, a_1, a_2, \dots$ is

$$A_n(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

eg. $A_4(x) = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \dots = 1 + 4x + 6x^2 + 4x^3 + x^4 = (1+x)^4$ Binomial Theorem

The Binomial Theorem $(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$ holds for all real values of m.

If m is a non-negative integer then $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is a non-negative integer (positive for $n = 0, 1, 2, \dots, m$; zero for $n > m$) in which case $(1+x)^m$ is a polynomial in x of degree m. This is a special case of the Binomial Series.

The Binomial coefficients are found by hand from Pascal's Triangle

0 1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

...

$\binom{m}{n}$ = entry n in row m of Pascal's Triangle

eg. $\binom{4}{2}$ = entry 2 in row 4

(start counting at 0, 1, 2, ...)

The recursive formula for generating Pascal's Triangle is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\begin{tikzcd}[column sep=small] \binom{n-1}{k-1} \arrow[dr] & & \binom{n-1}{k} \arrow[dl] \\ & \binom{n}{k} & \end{tikzcd}$$

Three proofs of Pascal's formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$:

<u>Combinatorial Proof</u> (counting proof): Consider the n-set $[n] = \{1, 2, \dots, n\}$.
Any k-subset $B \subseteq [n]$ is of one of the following two types:

(i) $n \in B$. In this case $B = \{n\} \cup B'$ where $B' \subseteq [n-1]$, $|B'| = k-1$.
There are $\binom{n-1}{k-1}$ ways to choose B' in this case.

(ii) $n \notin B$. In this case $B \subseteq [n-1]$. There are $\binom{n-1}{k}$ choices for B in this case.

The sum in cases (i) and (ii) must give $\binom{n}{k}$. $\square$

<u>Generating Function Proof</u>: Compare coefficients of x^k on both sides of

$$(1+x)^n = (1+x)(1+x)^{n-1}$$

$$1 + nx + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + x^n = (1 + x)\left(1 + (n-1)x + \dots \binom{n-1}{k-1}x^{k-1} + \binom{n-1}{k}x^k + \dots + x^{n-1}\right)$$

which gives $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. $\square$

Third Proof

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$n! = n \cdot (n-1)!$

$$= \frac{(n-1)!\,k}{(k-1)!\,(n-k)\,(n-k-1)!\,k} + \frac{(n-1)!\,(n-k)}{k\cdot(k-1)!\,(n-k-1)!\,(n-k)}$$

$$= \frac{(n-1)!\,k + (n-1)!\,(n-k)}{(k-1)!\,(n-k-1)!\,k\,(n-k)}$$

$$= \frac{n\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \quad \square$$

$$A_n(x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$ = the sum of the entries in row n of Pascal's triangle.

A combinatorial explanation for this result is
2^n = number of subsets of $[n]$ = $\sum_{i=0}^{n}$ (number of i-subsets of $[n]$) = $\sum_{i=0}^{n}\binom{n}{i}$

(or 2^n = number of bitstrings of length n which can be rewritten as $\sum_{i=0}^{n}\binom{n}{i}$ where $\binom{n}{i}$ is the number of bitstrings of length n having exactly i 1's.)