



Combinatorics

Book 3

Fourth method: Decompose $F(x) = \frac{1+x}{1-x-x^2}$ using partial fractions.

Factor the denominator $1-x-x^2 = (1-\alpha x)(1-\beta x)$

The roots are the same as the roots of x^2+x-1

$$\text{i.e. } \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

$$\text{The reciprocal roots are } \frac{2}{-1 \pm \sqrt{5}} \cdot \frac{-1 \mp \sqrt{5}}{-1 \mp \sqrt{5}} = \frac{2(-1 \mp \sqrt{5})}{1-5} = \frac{1 \mp \sqrt{5}}{2}$$

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618 \quad (\text{the golden ratio})$$

$$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Always use α, β in the algebraic simplification.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} = A \sum_{n=0}^{\infty} (\alpha x)^n + B \sum_{n=0}^{\infty} (\beta x)^n$$

$$= \sum_{n=0}^{\infty} \underbrace{(A\alpha^n + B\beta^n)}_{a_n} x^n$$

$a_n \sim A\alpha^n$ (exponential growth rate)

Note: The factors $1-\alpha x$, $1-\beta x$ reveal the reciprocal roots α, β . (The roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.)

$$\alpha + \beta = 1$$

$$\alpha - \beta = \sqrt{5}$$

$$\alpha\beta = -1$$

α, β are reciprocal roots of x^2+x-1

$$\frac{1}{\alpha^2} + \frac{1}{\alpha} - 1 = 0$$

$$1 + \alpha - \alpha^2 = 0$$

$$\alpha^2 = \alpha + 1$$

$$\beta^2 = \beta + 1$$

Use partial fractions to find A, B such that

$$\frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$\alpha^2 = \alpha + 1$$

$$\beta^2 = \beta + 1$$

$$1+x = A(1-\beta x) + B(1-\alpha x)$$

Evaluate at $x = \frac{1}{\alpha}$, then at $\frac{1}{\beta}$.

$$1 + \frac{1}{\alpha} = A(1 - \frac{\beta}{\alpha})$$

$$1 + \frac{1}{\beta} = B(1 - \frac{\alpha}{\beta})$$

$$\alpha^2 = \alpha + 1 = A(\alpha - \beta) = \sqrt{5}A \Rightarrow A = \frac{\alpha^2}{\sqrt{5}}$$

$$B = -\frac{\beta^2}{\sqrt{5}}$$

$\alpha \leftrightarrow \beta$ interchanged by algebraic conjugation

$$\sqrt{5} \leftrightarrow -\sqrt{5}$$

$$a_n = A\alpha^n + B\beta^n = \frac{\alpha^2}{\sqrt{5}}\alpha^n - \frac{\beta^2}{\sqrt{5}}\beta^n = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}$$

As $n \rightarrow \infty$, $\beta^n \rightarrow 0$ since $|\beta| < 1$ so $a_n \sim A\alpha^n$ where $A = \frac{\alpha^2}{\sqrt{5}}$.

Asymptotics If $f(n), g(n) \rightarrow \infty$ as $n \rightarrow \infty$, we write $f(n) \sim g(n)$ (f is asymptotic to g) if $\frac{f(n)}{g(n)} \rightarrow 1$. This is different from \approx (approximately equal).

eg. $\sqrt{n^2 + 10n} \rightarrow \infty$ as $n \rightarrow \infty$; $\sqrt{n^2 + 10n} \sim n$ as $n \rightarrow \infty$.

check: $\frac{\sqrt{n^2 + 10n}}{n} = \sqrt{1 + \frac{10}{n}} \rightarrow 1$. ($\lim_{n \rightarrow \infty} \sqrt{1 + \frac{10}{n}} = 1$).

$$\sqrt{n^2 + 10n} - n = \left(\sqrt{n^2 + 10n} - n\right) \cdot \frac{\sqrt{n^2 + 10n} + n}{\sqrt{n^2 + 10n} + n} = \frac{10n}{\sqrt{n^2 + 10n} + 10n} = \frac{10}{\sqrt{1 + \frac{10}{n}} + 1} \rightarrow 5 \quad \text{as } n \rightarrow \infty$$

$n^3 + 7n^2 \sim n^3$ as $n \rightarrow \infty$ Since $\frac{n^3 + 7n^2}{n^3} = 1 + \frac{7}{n} \rightarrow 1$ as $n \rightarrow \infty$
 yet $(n^3 + 7n^2) - n^3 = 7n^2 \rightarrow \infty$ as $n \rightarrow \infty$

In our case the convergence is stronger: not only is $a_n \sim A\alpha^n$ but moreover $a_n - A\alpha^n \rightarrow 0$. We can actually evaluate a_n by taking the closest integer to $A\alpha^n$.

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

Another example of partial fraction decomposition:

$$\begin{aligned} \frac{1+2x-3x^2}{1+x+4x^2+4x^3} &= \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{1+2x-3x^2}{(1+x)(1+2ix)(1-2ix)} = \frac{A}{1+x} + \frac{B}{1+2ix} + \frac{C}{1-2ix} \\ &= A \sum_{n=0}^{\infty} (-1)^n x^n + B \sum_{n=0}^{\infty} (2i)^n x^n + C \sum_{n=0}^{\infty} (2i)^n x^n = \sum_{n=0}^{\infty} \underbrace{(A(-1)^n + B(-2i)^n + C(2i)^n)}_{a_n} x^n \end{aligned}$$

OR

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+4x^2}$$

since $\frac{1}{1+x} = \frac{x}{1+4x^2} + \frac{1}{1+4x^2}$

How fast does a_n grow? The reciprocal roots of $1+x+4x^2+4x^3$ are $-1, -2i, 2i$. $| -1 | = 1$
 $| \pm 2i | = 2$.

$a_n \sim c2^n$. From Maple it seems $a_n \sim \frac{1}{10} 2^n$.

No!
 look again:

$$a_n \sim \begin{cases} \frac{9}{5} 2^n & \text{if } n \equiv 0 \pmod{4}; \\ \frac{1}{10} 2^n & \text{if } n \equiv 1 \pmod{4}; \\ -\frac{3}{5} 2^n & \text{if } n \equiv 2 \pmod{4}; \\ -\frac{1}{10} 2^n & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

$$F(x) = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+4x^2} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{9}{5}x + \frac{9}{5}}{1+4x^2} = -\frac{4}{5}(1-x+x^2-x^3+x^4-x^5+\dots) + \left(\frac{9}{5}x + \frac{9}{5}\right)(1-4x^2+16x^4-64x^6+\dots)$$

Solve for A, B, C

$$\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\dots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

$$\text{where } a_n = (-1)^{n+1} \frac{4}{5} + \begin{cases} \frac{9}{5}(-4)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{1}{5}(-4)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Alternatively,

(different constants A, B, C)

$$F(x) = \frac{A}{1+x} + \frac{B}{1+2ix} + \frac{C}{1-2ix} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{9}{5} - \frac{1}{10}i}{1+2ix} + \frac{\frac{9}{5} + \frac{1}{10}i}{1-2ix}$$

(Something like this... look at MAPLE session)

a_n "grows exponentially" (const. 2^n)

but $a_n \neq c2^n$. This happens because the denominator of $F(x)$ has two reciprocal roots of the same largest absolute value.

Another example in counting walks in a graph where this issue arises:



$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$w_n = w_n(1,1)$ = number of walks of length n from vertex 1 to itself.

n	0	1	2	3	4	5	6	...
w_n	1	0	2	0	4	0	8	...

$$W(x) = [I - xA]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - x \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2x \\ -x & 1 \end{bmatrix}^{-1} = \frac{1}{1-4x^2} \begin{bmatrix} 1 & 2x \\ x & 1 \end{bmatrix} = \begin{bmatrix} w_{11}(x) & w_{12}(x) \\ w_{21}(x) & w_{22}(x) \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$w(x) = w_{11}(x) = \frac{1}{1-4x^2} = 1 + 4x^2 + 16x^4 + 64x^6 + 256x^8 + \dots$$

$$w_n = w_n(r, 1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even.} \end{cases}$$

Denominator $1-4x^2 = (1+2x)(1-2x)$ has two ^{reciprocal} roots ± 2 having the same absolute value?

Remarks: $\frac{1}{1-4x^2}$ is preferred over $\frac{-\frac{1}{4}}{x^2 - \frac{1}{4}}$ since we want to use the geometric

series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$

Exponential growth $f(n) \sim ca^n$ (c, a, k constants)

Polynomial growth $f(n) \sim cn^k$ eg. $4n^3 + 7n^2 + 11n + 59 \sim 4n^3$

Other counting problems leading to a sequence where generating functions are used to express the solution:

Let a_n be the number of permutations of $[n] = \{1, 2, \dots, n\}$ (i.e. the number of ways I can list n students in order). Then $a_n = n!$. Its generating function is



$$F(x) = \sum_{n=0}^{\infty} n! x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + 120x^5 + 720x^6 + 5040x^7 + \dots$$

$$G(x) = \sum_{n=0}^{\infty} (n!)^2 x^n = 1 + x + 4x^2 + 36x^3 + 576x^4 + \dots$$