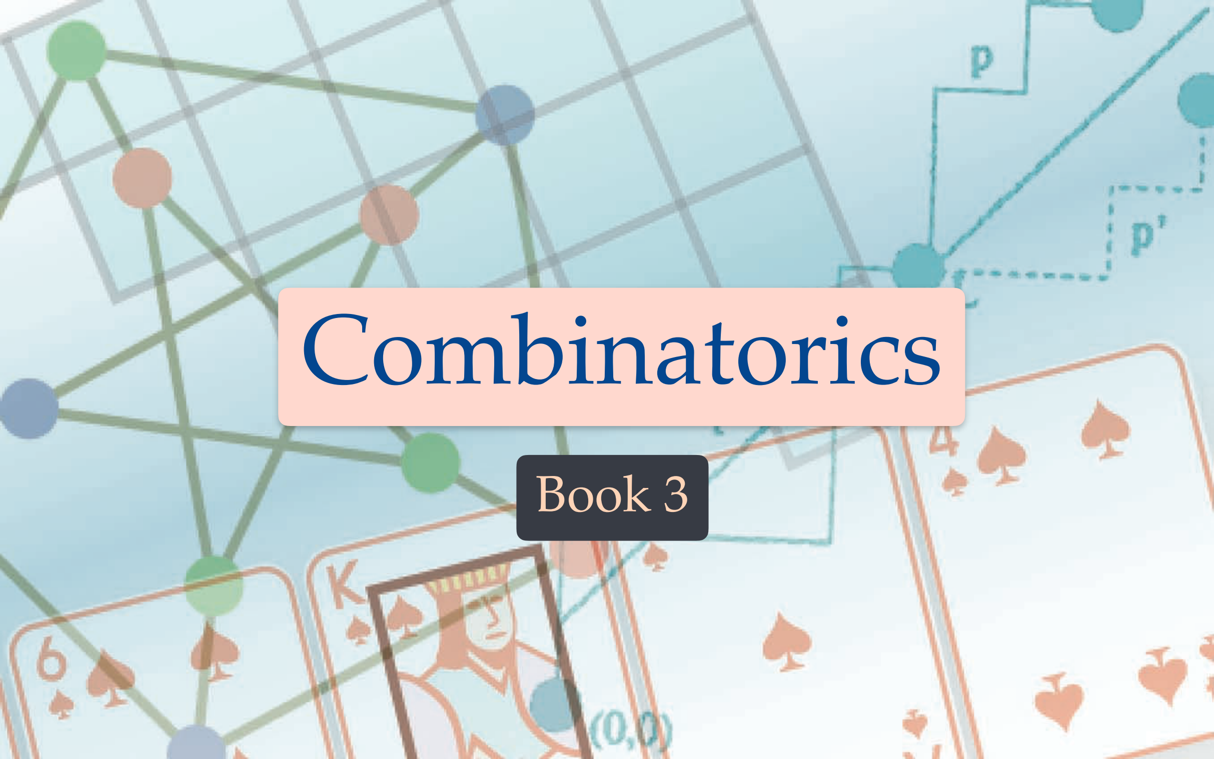


Combinatorics

Book 3

Fourth method: Decompose $F(x) = \frac{1+x}{1-x-x^2}$ using partial fractions.

Factor the denominator $1-x-x^2 = (1-\alpha x)(1-\beta x)$

The roots are the same as the roots of x^2+x-1

i.e. $\frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

Note: The factors $1-\alpha x$, $1-\beta x$ reveal the reciprocal roots α, β. (The roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.)

The reciprocal roots are $\frac{2}{-1 \pm \sqrt{5}} \cdot \frac{-1 \mp \sqrt{5}}{-1 \mp \sqrt{5}} = \frac{2(-1 \mp \sqrt{5})}{1-5} = \frac{1 \mp \sqrt{5}}{2}$

$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$ (the golden ratio)

$\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$

$\alpha + \beta = 1$

$\alpha - \beta = \sqrt{5}$

$\alpha\beta = -1$

α, β are reciprocal roots of x^2+x-1

$\frac{1}{\alpha^2} + \frac{1}{\alpha} - 1 = 0$

$1 + \alpha - \alpha^2 = 0$

$\alpha^2 = \alpha + 1$

$\beta^2 = \beta + 1$

Always use α, β in the algebraic simplification.

$$F(x) = \frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} . = A\sum_{n=0}^{\infty} (\alpha x)^n + B\sum_{n=0}^{\infty} (\beta x)^n$$

$$= \sum_{n=0}^{\infty} \boxed{\underbrace{(A\alpha^n + B\beta^n)}_{a_n}} x^n$$

$a_n \sim A\alpha^n$ (exponential growth rate)

Use partial fractions to find A, B such that

$$\frac{1+x}{1-x-x^2} = \frac{1+x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$\alpha^2 = \alpha + 1$
$\beta^2 = \beta + 1$

$$1+x = A(1-\beta x) + B(1-\alpha x)$$

Evaluate at $x = \frac{1}{\alpha}$, then at $\frac{1}{\beta}$.

$$1 + \frac{1}{\alpha} = A\left(1 - \frac{\beta}{\alpha}\right) \qquad 1 + \frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right)$$

$\alpha \longleftrightarrow \beta$ interchanged by algebraic conjugation

$$\alpha^2 = \alpha + 1 = A(\alpha - \beta) = \sqrt{5}A \implies A = \frac{\alpha^2}{\sqrt{5}} \qquad B = -\frac{\beta^2}{\sqrt{5}}$$

$\sqrt{5} \longleftrightarrow -\sqrt{5}$

$$a_n = A\alpha^n + B\beta^n = \frac{\alpha^2}{\sqrt{5}}\alpha^n - \frac{\beta^2}{\sqrt{5}}\beta^n = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\sqrt{5}}$$

As $n \to \infty$, $\beta^n \to 0$ since $|\beta| < 1$ so $a_n \sim A\alpha^n$ where $A = \frac{\alpha^2}{\sqrt{5}}$.

<u>Asymptotics</u> If $f(n), g(n) \to \infty$ as $n \to \infty$, we write $f(n) \sim g(n)$ (f is <u>asymptotic</u> to g) if $\frac{f(n)}{g(n)} \to 1$. This is different from "$\approx$". (approximately equal).

eg. $\sqrt{n^2 + 10n} \to \infty$ as $n \to \infty$; $\sqrt{n^2+10n} \sim n$ as $n \to \infty$.

Check: $\frac{\sqrt{n^2+10n}}{n} = \sqrt{1 + \frac{10}{n}} \to 1$. $\left(\lim_{n\to\infty} \sqrt{1+\frac{10}{n}} = 1\right)$.

$$\sqrt{n^2+10n} - n = \left(\sqrt{n^2+10n} - n\right) \cdot \frac{\sqrt{n^2+10n} + n}{\sqrt{n^2+10n} + n} = \frac{10n}{\sqrt{n^2+10n} + 10n} = \frac{10}{\sqrt{1+\frac{10}{n}} + 1} \to 5 \quad \text{as } n \to \infty.$$

$n^3+7n^2 \sim n^3$ as $n\to\infty$ Since $\frac{n^3+7n^2}{n^3} = 1+\frac{7}{n} \to 1$ as $n\to\infty$

yet $(n^3+7n^2)-n^3 = 7n^2 \to \infty$ as $n\to\infty$

In our case the convergence is stronger: not only is $a_n \sim A\alpha^n$ but moreover $a_n - A\alpha^n \to 0$. We can actually evaluate a_n by taking the closest integer to $A\alpha^n$.

$$\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$$

Another example of partial fraction decomposition:

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{1+2x-3x^2}{(1+x)(1+2ix)(1-2ix)} = \frac{A}{1+x}+\frac{B}{1+2ix}+\frac{C}{1-2ix}$$

$$= A\sum_{n=0}^{\infty}(-1)^n x^n + B\sum_{n=0}^{\infty}(-2i)^n x^n + C\sum_{n=0}^{\infty}(2i)^n x^n = \sum_{n=0}^{\infty}\Big[\underbrace{A(-1)^n+B(-2i)^n+C(2i)^n}_{a_n}\Big]x^n$$

<u>OR</u>

$$\frac{1+2x-3x^2}{1+x+4x^2+4x^3} = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x}+\frac{Bx+C}{1+4x^2}$$

since $\frac{1}{1+x}, \frac{x}{1+4x^2}, \frac{1}{1+4x^2}$

How fast does a_n grow? The reciprocal roots of $1+x+4x^2+4x^3$ are $-1, -2i, 2i$. $|-1|=1$, $|\pm 2i| = 2$.

$a_n \sim c\,2^n$. From Maple it seems $a_n \sim \frac{1}{10}2^n$.

No! look again:

$$a_n \sim \begin{cases} \frac{9}{5}\cdot 2^n, & \text{if } n\equiv 0 \bmod 4;\\ \frac{1}{10}2^n, & \text{if } n\equiv 1 \bmod 4;\\ -\frac{9}{5}2^n, & \text{if } n\equiv 2 \bmod 4;\\ -\frac{1}{10}\cdot 2^n, & \text{if } n\equiv 3 \bmod 4.\end{cases}$$

$$F(x) = \frac{1+2x-3x^2}{(1+x)(1+4x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+4x^2} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{1}{5}x + \frac{9}{5}}{1+4x^2} = -\frac{4}{5}\left(1-x+x^2-x^3+x^4-x^5+\cdots\right)$$
$$+\left(\tfrac{1}{5}x+\tfrac{9}{5}\right)\left(1-4x^2+16x^4-64x^6+\cdots\right)$$

Solve for A, B, C

$$\frac{1}{1-u} = 1+u+u^2+u^3+u^4+\cdots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

where $a_n = (-1)^{n+1}\frac{4}{5} + \begin{cases} \frac{9}{5}(-4)^{\frac{n}{2}} & \text{if } n \text{ is even} \\ \frac{1}{5}(-4)^{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$

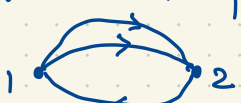
Alternatively, (different constants A, B, C)

$$F(x) = \frac{A}{1+x} + \frac{B}{1+2ix} + \frac{C}{1-2ix} = \frac{-\frac{4}{5}}{1+x} + \frac{\frac{9}{5}-\frac{1}{10}i}{1+2ix} + \frac{\frac{9}{5}+\frac{1}{10}i}{1-2ix}$$

(Something like this... look at MAPLE session)

$|a_n|$ "grows exponentially" $(\text{const}\cdot 2^n)$ but $a_n \not\sim c2^n$. This happens because the denominator of $F(x)$ has two reciprocal roots of the same largest absolute value.

Another example in counting walks in a graph where this issue arises:



$A: \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$w_n = w_n(1,1) =$ number of walks of length n from vertex 1 to itself.

n	0	1	2	3	4	5	6	$\cdots$
w_n	1	0	2	0	4	0	8	$\cdots$

$$W(x) = [I - xA]^{-1} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - x\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} 1 & -2x \\ -x & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{1-4x^2}\begin{bmatrix} 1 & 2x \\ x & 1 \end{bmatrix} = \begin{bmatrix} W_{11}(x) & W_{12}(x) \\ W_{21}(x) & W_{22}(x) \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$w(x) = w_{11}(x) = \frac{1}{1-4x^2} = 1+4x^2+16x^4+64x^6+256x^8+\cdots$$

$$w_n = w_n(1,1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even.} \end{cases}$$

Denominator $1-4x^2 = (1+2x)(1-2x)$ has two $\left\{\begin{array}{l}\text{reciprocal} \\ \text{roots}\end{array}\right.$ ± 2 having the same absolute value 2.

<u>Remarks</u>: $\frac{1}{1-4x^2}$ is preferred over $\frac{-\frac{1}{4}}{x^2-\frac{1}{4}}$ since we want to use the geometric series $\frac{1}{1-u} = 1+u+u^2+u^3+\cdots$

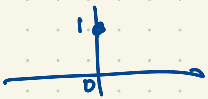
Exponential growth $f(n) \sim c a^n$ $\quad$ (c, a, k constants)

Polynomial growth $f(n) \sim c n^k$

eg. $4n^3+7n^2+11n+59 \sim 4n^3$

Other counting problems leading to a sequence where generating functions are used to express the solution:

Let a_n be the number of permutations of $[n] = \{1, 2, \ldots, n\}$ (i.e. the number of ways I can list n students in order). Then $a_n = n!$. Its generating function is



$$F(x) = \sum_{n=0}^{\infty} n!\, x^n = 1+x+2x^2+6x^3+24x^4+120x^5+720x^6+5040x^7+\cdots$$

$$G(x) = \sum_{n=0}^{\infty} (n!)^2 x^n = 1+x+4x^2+36x^3+576x^4+\cdots$$

$\binom{n}{k}$ is the number of k-subsets of an n-set

i.e. the number of bitstrings of length n having k 1's (and $n-k$ zeroes).

If $a_k = \binom{n}{k}$ where n is fixed then the generating function for the sequence $a_0, a_1, a_2, \dots$ is

$$A_n(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

eg. $A_4(x) = \binom{4}{0} + \binom{4}{1}x + \binom{4}{2}x^2 + \dots = 1 + 4x + 6x^2 + 4x^3 + x^4 = (1+x)^4$ Binomial Theorem

The Binomial Theorem $(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$ holds for all real values of m.

If m is a non-negative integer then $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ is a non-negative integer (positive for $n = 0, 1, 2, \dots, m$; zero for $n > m$) in which case $(1+x)^m$ is a polynomial in x of degree m. This is a special case of the Binomial Series.

The Binomial coefficients are found by hand from Pascal's Triangle

0 1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

...

$\binom{m}{n}$ = entry n in row m of Pascal's Triangle

eg. $\binom{4}{2}$ = entry 2 in row 4

(start counting at 0, 1, 2, ...)

The recursive formula for generating Pascal's Triangle is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\begin{tikzcd}[column sep=small] \binom{n-1}{k-1} \arrow[dr] & & \binom{n-1}{k} \arrow[dl] \\ & \binom{n}{k} & \end{tikzcd}$$

Three proofs of Pascal's formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$:

Combinatorial Proof (counting proof): Consider the n-set $[n] = \{1, 2, \ldots, n\}$.

Any k-subset $B \subseteq [n]$ is of one of the following two types:

(i) $n \in B$. In this case $B = \{n\} \cup B'$ where $B' \subseteq [n-1]$, $|B'| = k-1$.

There are $\binom{n-1}{k-1}$ ways to choose B' in this case.

(ii) $n \notin B$. In this case $B \subseteq [n-1]$. There are $\binom{n-1}{k}$ choices for B in this case.

The sum in cases (i) and (ii) must give $\binom{n}{k}$. $\square$

Generating Function Proof: Compare coefficients of x^k on both sides of

$$(1+x)^n = (1+x)(1+x)^{n-1}$$

$$1 + nx + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + x^n = (1 + x)\left(1 + (n-1)x + \cdots \binom{n-1}{k-1}x^{k-1} + \binom{n-1}{k}x^k + \cdots + x^{n-1}\right)$$

which gives $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. $\square$

Third Proof

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$n! = n \cdot (n-1)!$

$$= \frac{(n-1)!\,k}{(k-1)!\,(n-k)\,(n-k-1)!\,k} + \frac{(n-1)!\,(n-k)}{k \cdot (k-1)!\,(n-k-1)!\,(n-k)}$$

$$= \frac{(n-1)!\,k + (n-1)!\,(n-k)}{(k-1)!\,(n-k-1)!\,k\,(n-k)}$$

$$= \frac{n\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}. \quad \square$$

$$A_n(x) = (1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

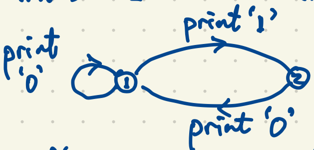
$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$ = the sum of the entries in row n of Pascal's triangle.

A combinatorial explanation for this result is

2^n = number of subsets of $[n]$ = $\sum_{i=0}^{n}$ (number of i-subsets of $[n]$) = $\sum_{i=0}^{n} \binom{n}{i}$

(or 2^n = number of bitstrings of length n which can be rewritten as $\sum_{i=0}^{n} \binom{n}{i}$ where $\binom{n}{i}$ is the number of bitstrings of length n having exactly i 1's.)

HW3 #2 is similar to the example on the handout on Fibonacci numbers



$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

This directed graph is an example of a nondeterministic finite automaton with two states ①, ②.

How many walks are there starting at vertex 1? $w_n = w_n(1,1) + w_n(1,2)$

Printout 0101000 represents the walk $(1,2,1,2,1,1,1,1)$ of length 7

The walks of length n starting at vertex 1 are in one-to-one correspondence with 11-free bitstrings of length n.

More generally, many counting problems (where recursion plays a role) are equivalent to counting walks in graphs.

Recall: Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ where $\binom{n}{k}$ (binomial coefficient "n choose k") equals the number of k-subsets of an n-set. ($n \geq 0$ integer)

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } k \in \{0,1,2,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$$

Multinomial Theorem $(x_1 + x_2 + \dots + x_r)^n = \sum_{i_1,\dots,i_r} \binom{n}{i_1, i_2, \dots, i_r} x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r}$

$$\binom{n}{i_1, i_2, \dots, i_r} = \frac{n!}{i_1!\, i_2! \cdots i_r!} \quad \text{if } i_1,\dots,i_r \geq 0,\ i_1 + \dots + i_r = n \ ; \quad 0 \text{ otherwise}$$

Multinomial Coefficient

eg. $(x+y+z)^3 = \sum_{\substack{i+j+k=3 \\ i,j,k\geq 0}} \binom{3}{i,j,k} x^i y^j z^k = x^3+y^3+z^3 + 3x^2y + 3xy^2 + 3x^2z + 3xz^2 + 3y^2z + 3yz^2$
$+ 6xyz$

(Trinomial expansion)

$\binom{3}{3,0,0} = \frac{3!}{3!\,0!\,0!} = \frac{6}{6\cdot 1\cdot 1} = 1 = \binom{3}{0,3,0} = \binom{3}{0,0,3}$

$\binom{3}{2,1,0} = \frac{6}{2\cdot 1\cdot 1} = 3 = \binom{3}{0,2,1} = \cdots$

$\binom{3}{1,1,1} = \frac{3!}{1!\,1!\,1!} = \frac{6}{1\cdot 1\cdot 1} = 6$

Check: $3^3 = \underbrace{1+1+1}_{3} + \underbrace{3+\cdots+3}_{6} + 6 = 27$

(evaluating at $(1,1,1)$).

How many words can be formed by permuting the letters of MISSISSIPPI ?
(Words are strings of letters where the order is important.)

$\frac{11!}{1!\,4!\,4!\,2!} = \binom{11}{1,4,4,2} = 34{,}650$.

How many words can be formed by permuting the bits in 01110010010 ?

$\binom{11}{5,6} = \frac{11!}{5!\,6!} = \binom{11}{5} = \binom{11}{6} = 462$

Say M&M's are made in 6 different colors. How many different ways can we have a handful of 10 M&Ms? or n M&M's ?

a_n = number of ways to have a handful of n M&Ms ?

n	0	1	2	$\cdots$
a_n	1	6	21	$\cdots$

If M&M's come in the colors red "R", blue "Bl", green "G", orange "O", yellow "Y", brown "Br", then there are $\binom{15}{5}$ ways to draw a handful of ten M&M's e.g.

"X" is a divider

R R X X G X O O O O X Y X Br Br

$\underbrace{* \; *}_{\text{red}} \; X \underset{\text{blue}}{} X \underbrace{*}_{\text{green}} X \underbrace{* * * *}_{\text{orange}} X \underbrace{*}_{\text{yellow}} X \underbrace{* \; *}_{\text{brown}}$ represents the color distribution

2 red
0 blue
1 green
4 orange
1 yellow
2 brown

10 M&M's

The possible color distributions for a handful of 10 M&M's are in one-to-one correspondence with the number of words of length 15 over a binary alphabet '*', 'X'. So the number of handfuls of 10 M&M's which come in 6 colors is $\binom{15}{5}$.

If M&M's come in k colors and we select n M&M's from this batch, the number of possible color distributions is $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

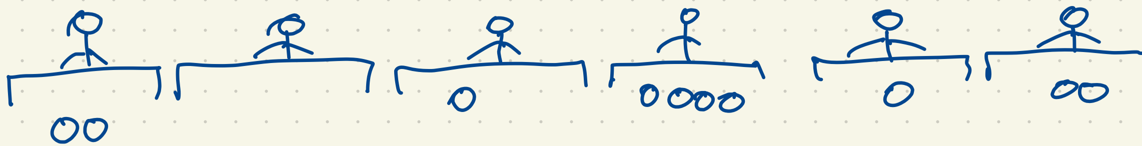
① Suppose I want to hand out n books (all different) to k students. How many ways can I do this?

$$\underbrace{k \times k \times \cdots \times k}_{n \text{ times}} = k^n \text{ choices.}$$

$\binom{n+k-1}{n}$

② How many ways can I hand out n identical silver dollars to k students?

Eg. I hand out 10 identical silver dollars to 6 students.



$$\longleftrightarrow \quad OO \,|\,|\, O \,|\, OOOO \,|\, O \,|\, OO$$

Answer: $\binom{15}{5} = \binom{15}{10}$

Note: Problem ① is counting functions $[n] \longrightarrow [k]$.

In Problem ②, what if we require each student to get at least one of the silver dollars? Instead of $\binom{10+6-1}{10}$, the answer is $\binom{4+6-1}{4} = \binom{9}{4}$.

Suppose I want to hand out k different books to n students, in such a way that each student gets at most one book. How many ways can we distribute the books?

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

This equals zero if $k > n$.

no. of choices of student to give book 1 to — 2nd book — 3rd — k^{th} book

$P(n,k) = 0$ if $k < n$

$P(n,k) = n!$ if $k = n$

$P(n,k)$ is also denoted $n_{(k)}$ or various other notations

("descending factorial" or "falling faction")

$P(n,k)$ is the number of one-to-one maps $[k] \longrightarrow [n]$ (injections)

Question: How many surjections $[k] \longrightarrow [n]$? (functions that are onto, i.e. how many ways can we hand out k different books to n students if we want every student to get at least one book)?

Binomial Theorem $(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

What if m is not an integer?

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!} = \frac{m\cdot(m-1)(m-2)\cdots(m-k+1)\cancel{(m-k)(m-k-1)\cdots 1}}{k!\;\cancel{(m-k)(m-k-1)(m-k-2)\cdots 1}} = \frac{P(m,k)}{k!}$$

$P(m,k) = m(m-1)(m-2)\cdots(m-k+1)$ is defined for all $k \in \{0,1,2,3,4,\cdots\}$ and m any real number.

$P(m,0) = 1$

$P(m,1) = m$

$P(m,2) = m(m-1)$

eg. $$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k = 1 + \underbrace{\frac{\frac{1}{2}}{1!}}_{P(\frac{1}{2},1)} x + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}}_{P(\frac{1}{2},2)} x^2 + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}}_{P(\frac{1}{2},3)} x^3 + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}}_{P(\frac{1}{2},4)} x^4 + \cdots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

Suppose I want to give out n identical silver dollars to 3 students x, y, z. How many ways can I do this? This is the same as counting bitstrings of length $n+2$ having 2 ones and n zeroes e.g.

$$\underbrace{x^2 y z^4}_{} \longleftrightarrow \underbrace{00}_{x}1\underbrace{0}_{y}1\underbrace{0000}_{z}$$ represents one way to distribute 7 silver dollars to x, y, z

$$\binom{9}{2} = \frac{9 \cdot 8}{2 \cdot 1} \quad \begin{matrix} \leftarrow P(9,2) \\ \leftarrow 2! \end{matrix} \quad = 36 \text{ ways to distribute 7 identical silver dollars to 3 students}$$

Expand

$$\frac{1}{(1-x)(1-y)(1-z)} = (1+x+x^2+x^3+\cdots)(1+y+y^2+y^3+\cdots)(1+z+z^2+z^3+\cdots)$$

$$= \underbrace{1}_{\text{degree } 0} + \underbrace{x+y+z}_{\text{degree } 1} + \underbrace{x^2+y^2+z^2+xy+xz+yz}_{\text{degree } 2} + \underbrace{x^3+y^3+z^3+x^2y+xy^2+x^2z+xz^2+y^2z+yz^2+xyz}_{\text{degree } 3} + \cdots$$

The term $x^i y^j z^k$ of degree $i+j+k$ represents how we can give i coins to x, j coins to y, k coins to z.

The number of ways to distribute n coins to 3 students is the number of terms of degree n in our expansion. To isolate terms of degree n in the expansion, do the following: replace x, y, z by tx, ty, tz.

$$\frac{1}{(1-tx)(1-ty)(1-tz)} = 1 + t(x+y+z) + t^2(x^2+y^2+z^2+xy+xz+yz) + t^3(x^3+y^3+\cdots+xyz) + \cdots$$

The coefficient of t^n in this series gives all the ways to distribute n coins to three students x, y, z. The number of ways to distribute n coins to 3 students, replace x, y, z by 1.

$$\frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + \cdots$$

For this we can use the Binomial Theorem.

How many ways can we distribute n identical silver dollars to k students?

Call the students $x_1, x_2, \cdots, x_k$.

$$\prod_{i=1}^{k} \frac{1}{1-x_i} = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_k)} = \prod_{i=1}^{k}(1+x_i+x_i^2+x_i^3+\cdots) = \sum_{i_1,\cdots,i_k\geq 0} x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$$

In order to collect terms of each degree $n\geq 0$, replace $x_1,\cdots,x_k$ by $tx_1,\cdots,tx_k$:

$$\prod_{i=1}^{k}\frac{1}{1-tx_i} = \sum_{i_1,\cdots,i_k\geq 0} t^{i_1+i_2+\cdots+i_k} x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$$

Now replace $x_1,\cdots,x_k$ by 1. $\prod_{i=1}^{k}\frac{1}{1-t} = \frac{1}{(1-t)^k} = \sum_{n=0}^{\infty}(\quad)t^n$

number of monomials $x_1^{i_1}\cdots x_k^{i_k}$ of degree $i_1+i_2+\cdots+i_k=n$

= number of solutions of

$i_1+i_2+\cdots+i_k=n$

$(i_1,\cdots,i_k\geq 0)$

= number of ways to give

i_1 coins to x_1,

i_2 " " x_2,

$\vdots$

i_k ----- x_k.

$$\frac{1}{(1-t)^k} = (1-t)^{-k} = \sum_{r=0}^{\infty}\binom{-k}{r}(-t)^r = \sum_{r=0}^{\infty}\frac{(-k)(-k-1)(-k-2)\cdots(-k-r+1)}{r!}(-1)^r t^r$$
$$= (1+(-t))^{-k}$$

$$= \sum_{r=0}^{\infty}\cancel{(-1)^r}\frac{k(k+1)(k+2)\cdots(k+r-1)}{r!}\cancel{(-1)^r}t^r = \sum_{r=0}^{\infty}\underbrace{\frac{P(k+r-1,r)}{r!}}_{\binom{k+r-1}{r}}t^r = \sum_{n=0}^{\infty}\binom{n+k-1}{n}t^n$$

Thus the number of ways to give k identical coins to k students is $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.

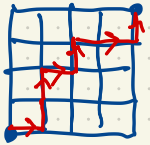
Number of ways to distribute 7 identical coins to 3 students is the coefficient of t^7 in

$$\frac{1}{(1-t)^3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 28t^6 + 36t^7 + \cdots \qquad \binom{9}{2} = 36$$

The sequence of coefficients is $\binom{n}{2} = 1, 3, 6, 10, 15, 21, 28, 36, \cdots$ is the triangular numbers
$$= 1 + 2 + 3 + \cdots + n$$

1 3 6 10

In a city downtown, all streets run north-south and east-west, forming a grid. How many ways can you travel from one intersection to another intersection that is n blocks north and n blocks east if we require a path of shortest distance (2n blocks)?



ENNENEEEN

e.g. $\binom{8}{4} = 70$ words of length 8 over the binary alphabet $\{E, N\}$

There are $\binom{2n}{n}$ shortest paths in the city grid to walk n blocks north and n blocks east.

This gives a sequence 1, 2, 6, 20, 70, ...

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

What is the generating function for this problem?

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots$$

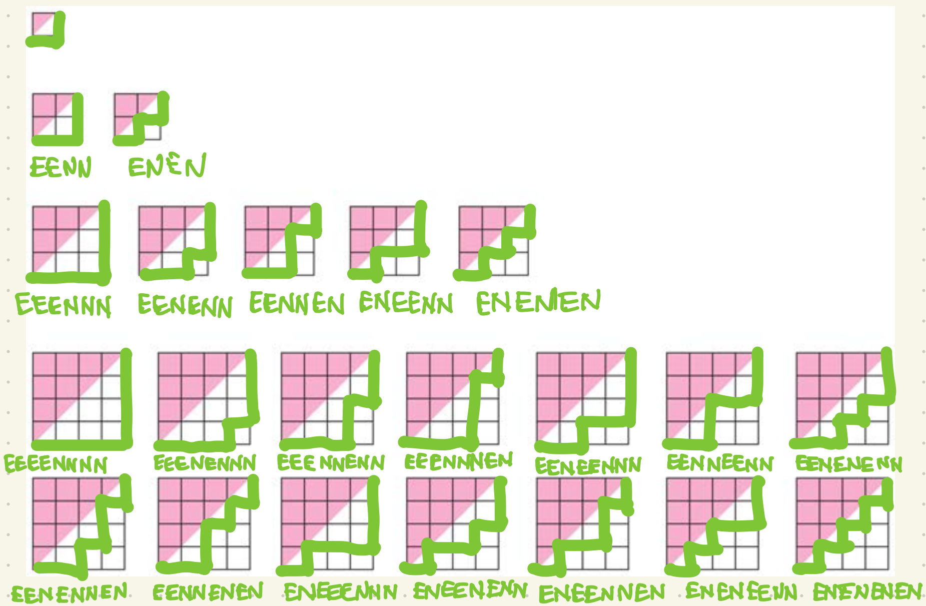
This is a warmup to our next problem. In both cases we can use the Binomial Theorem.

$$A(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \sum_{n=0}^{\infty} \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot (2n)}{1\cdot 2\cdot 3\cdot \cdots \cdot n \cdot 1\cdot 2\cdot 3\cdot \cdots \cdot n} x^n = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)\cdot 2\cdot 4\cdot 6\cdots \cdot 2n}{1\cdot 2\cdot 3\cdots n \cdot 1\cdot 2\cdot 3\cdot \cdots \cdot n} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots (2n-1)}{n!} 2^n x^n = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}(-4)^n x^n$$

$$= \sum_{n=0}^{\infty} \frac{P\left(-\frac{1}{2}, n\right)}{n!} (-4x)^n = \left(1+(-4x)\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots$$

This time count shortest paths (distance $2n$) in a city grid where we must walk n blocks north and n blocks east without going above the main diagonal "$y=x$":



C_n = number of solutions

n	0	1	2	3	4
C_n	1	1	2	5	14

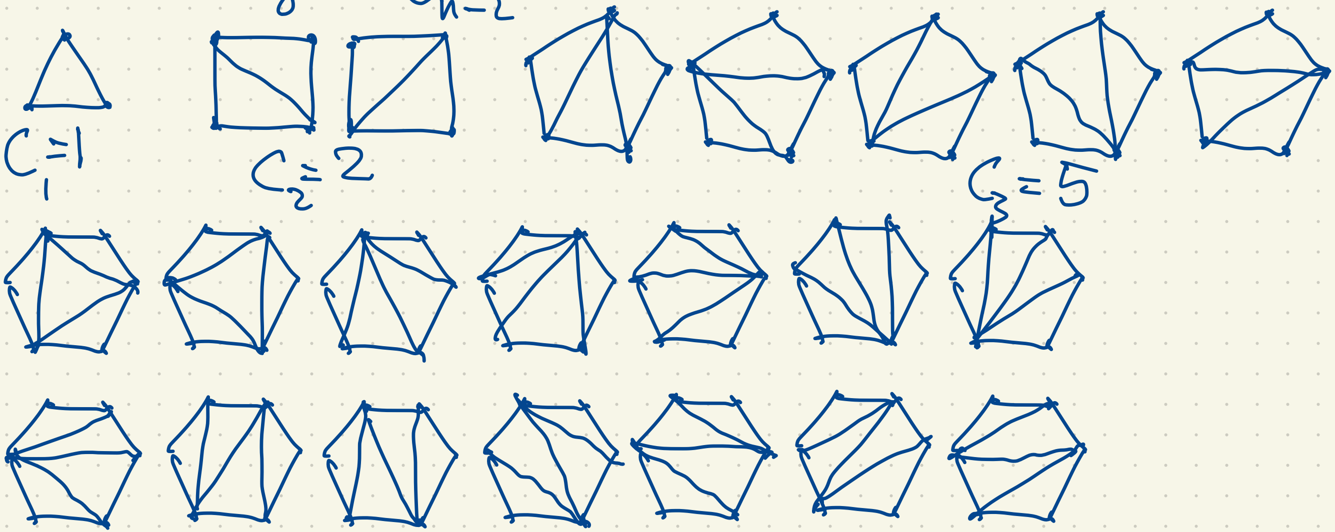
C_n is the n^{th} Catalan number.

C_n is the number of Dyck paths of length $2n$ defined above.

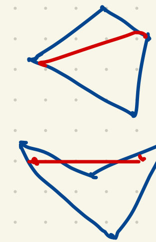
After observing $C_0 = 1$, we need a recurrence formula for C_n, for $n \geq 1$.

The Catalan numbers arise in many contexts.

Eg. How many ways can we join vertices of a convex n-gon to form a subdivision into n-2 triangles? C_{n-2}



$C_1 = 1$ $\quad C_2 = 2$ $\quad C_3 = 5$



convex 4-gon

non-convex 4-gon

$C_4 = 14$

Consider a product of n factors $u_1 u_2 \cdots u_n$ which is to be evaluated by multiplying 2 at a time. How many ways can the product be parenthesized to achieve the answer? C_{n-1}

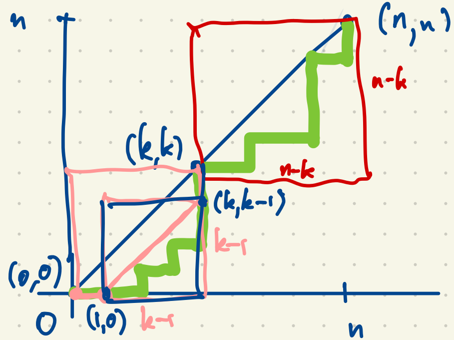
$n=1$: (a) $\quad C_0 = 1$ way

$n=2$: (ab) $\quad C_1 = 1$ way

$n=3$: $(ab)c$, $a(bc)$ $\quad C_2 = 2$ ways

$n=4$: $(ab)(cd)$, $((ab)c)d$, $(a(bc))d$, $a((bc)d)$, $a(b(cd))$ $\quad C_3 = 5$ ways

Recurrence formula for C_n, $n \geq 1$ (number of Dyck paths)



Let $k \in \{1,2,\cdots,n\}$ be the first value for which the Dyck path returns to the line $y=x$ i.e. $k \in \{1,2,\cdots,n\}$ is the smallest number for which (k,k) is in the Dyck path.

The number of choices for the portion of the Dyck path from (k,k) to (n,n) is C_{n-k}.

The number of choices for the portion of the path from $(0,0)$ to (k,k) is not exactly C_k since that would include paths that possibly hit the line $y=x$ before (k,k). The first portion of the Dyck path consists of: one block east, then a Dyck path from $(1,0)$ to $(k,k-1)$, then one block north. There are C_{k-1} such Dyck paths in this $(k-1)\times(k-1)$ square.

So $C_n = \sum_{k=1}^{n} C_{k-1} C_{n-k}$ i.e.

$$C_1 = C_0 C_0 = 1\cdot 1 = 1$$

$$C_2 = C_0 C_1 + C_1 C_0 = 1\cdot 1 + 1\cdot 1 = 2$$

$$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0 = 1\cdot 2 + 1\cdot 1 + 2\cdot 1 = 5$$

$$C_4 = C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0 = 5\cdot 1 + 2\cdot 1 + 1\cdot 2 + 1\cdot 5 = 14$$

The generating function for C_n is

$$C(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$$

satisfies

$$C(x)^2 = (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots)(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots)$$

$$= \underbrace{C_0 C_0}_{C_1} + \underbrace{(C_0 C_1 + C_1 C_0)}_{C_2} x + \underbrace{(C_0 C_2 + C_1 C_1 + C_2 C_0)}_{C_3} x^2 + \underbrace{(C_0 C_3 + C_1 C_2 + C_2 C_1 + C_3 C_0)}_{C_4} x^3 + \cdots$$

$$1 + x\, C(x)^2 = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots = C(x)$$

$$\boxed{x\, C(x)^2 - C(x) + 1 = 0} \implies C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm (1 - 2x - 2x^2 - 4x^3 - 10x^4 - \cdots)}{2x}$$

With the '+' sign, $\frac{2 - 2x - 2x^2 - 4x^3 - 10x^4 - \cdots}{2x} = \frac{1}{x} - 1 - x - 2x^2 - 5x^3 - \cdots$

So we must use the '−' sign: $C(x) = \frac{2x + 2x^2 + 4x^3 + 10x^4 + \cdots}{2x} = 1 + x + 2x^2 + 5x^3 + \cdots$

$C_0 \quad C_1 = 1 \quad C_2 \quad C_3$

Compare:

$$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum \left(\sum_{k=0}^{n} a_{n-k} b_k\right) x^n$$

convolution of the two sequences a_n, b_n

$(f * g)(x) = \int f(x-t) g(t)\, dt$ is the convolution of f, g

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - \sum_{k=0}^{\infty}\binom{1/2}{k}(-4x)^k}{2x} = -\frac{1}{2x}\sum_{k=1}^{\infty}\binom{1/2}{k}(-4x)^k$$

$$= -\frac{1}{2x}\sum_{k=1}^{\infty}\frac{\frac12(\frac12-1)(\frac12-2)(\frac12-3)\cdots(\frac12-(k-1))}{k!}(-4x)^k = -\frac{1}{2x}\sum_{k=1}^{\infty}\frac{\frac12(-\frac12)(-\frac32)(-\frac52)\cdots(-\frac{2k-3}{2})}{k!}(-4x)^k$$

$n = k-1$

$k = n+1$

$$= -\frac{1}{2x}\sum_{n=0}^{\infty}\frac{\frac12\overbrace{(-\tfrac12)(-\tfrac32)(-\tfrac52)\cdots(-\tfrac{2n-1}{2})}^{n \text{ factors}}}{(n+1)!}\underbrace{(-4x)^{n+1}}_{n+1} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\frac12\cdot\frac32\cdot\frac52\cdots\frac{2n-1}{2}}{(n+1)!}4^{n+1}x^n$$

There are 2n+2 minus signs which cancel.

I'm ~~off~~ by a ~~factor~~ of 2

Look at handout for correct derivation

$$= \sum_{n=0}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{(n+1)!\,2^{n+2}}\cdot 2^{2n+2}x^n = \sum_{n=0}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{(n+1)!}\cdot 2^{n+1}x^n = \sum_{n=0}^{\infty}\frac{1\cdot3\cdot5\cdots(2n-1)}{(n+1)!}\cdot\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot4\cdot6\cdots(2n)}2^{n+1}x^n$$

$$= \sum_{n=0}^{\infty}\frac{(2n)!}{(n+1)!\,\cancel{2^n}\cdot1\cdot2\cdot3\cdots n}\cancel{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(2n)!}{(n+1)!\,n!}x^n = \sum_{n=0}^{\infty}\frac{(2n)!}{(n+1)n!\,n!}x^n = \sum_{n=0}^{\infty}\underbrace{\frac{1}{n+1}\binom{2n}{n}}_{C_n}x^n$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad \text{eg. } C_4 = \frac15\binom84 = \frac{8\cdot7\cdot6\cdot5}{5\cdot4\cdot3\cdot2\cdot1} = 14.$$

Note: $C(x)$ is not a rational function. It is an algebraic function

How many ways can a cashier return 83 cents in change to a customer using pennies, nickels, dimes, and quarters? (Any two pennies are identical; similarly for nickels, dimes, quarters).

The generating function $F(x) = \dfrac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}$ counts the number of ways to make n cents into change.

$$F(x) = (1+x+x^2+x^3+x^4+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)(1+x^{25}+x^{50}+x^{75}+\cdots)$$

$$= \sum_{p,n,d,q=0}^{\infty} x^p \cdot x^{5n} \cdot x^{10d} \cdot x^{25q} = \sum_{p,n,d,q=0}^{\infty} x^{p+5n+10d+25q} = \sum_{k=0}^{\infty} \Big(\qquad \Big) x^k$$

number of ways to write k as $p+5n+10d+25q$ where $p,n,d,q \geq 0$

= number of ways to make k cents in change using pennies, nickels, dimes, quarters.

How many ways can we place k indistinguishable (identical) objects in n unmarked (identical) envelopes?

Warm-up: How many ways can n identical silver dollars be divided into nonempty piles?

Say $n=6$: $6 = 5+1 = 4+2 = 4+1+1 = \boxed{3+3} = \boxed{3+2+1} = \boxed{3+1+1+1}$

$= 2+2+2 = 2+2+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1$

$p(n)$ = number of partitions of n = number of ways to write n as a sum of positive integers if the order of the terms doesn't matter

$p(6) = 11$. The 11 partitions of 6 are $(6), (5,1), (4,1,1), \ldots, (1,1,1,1,1,1)$.

By convention we list terms of each partition in weakly decreasing order: $(n_1, n_2, \ldots, n_k)$ is a partition of n if $n_1+n_2+\cdots+n_k = n$, each n_i is a positive integer, and $n_1 \geq n_2 \geq n_3 \geq \cdots \geq n_k$. We write $6 \vdash (4,1,1)$ for example.

The generating function for $p(n)$ is $g(x) = \dfrac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}$ (infinite product)

$$g(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots = \sum_{n=0}^{\infty} p(n)\, x^n$$

why? The coefficient of x^n in

$$g(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)(1+x^4+x^8+x^{12}+\cdots) \times \cdots$$

$$= \sum_{n_1, n_2, n_3, n_4, \ldots \geq 0} x^{n_1} \cdot x^{2n_2} \cdot x^{3n_3} \cdot x^{4n_4} \cdots = \sum_{n=0}^{\infty} \left(p(n) \right) x^n$$

number of ways to write n as a sum of n_1 ones, n_2 twos, n_3 threes, etc.

$P_k(n)$ = number of ways to put n silver dollars in k nonempty piles or k unmarked envelopes where the order of the piles doesn't matter.

= number of partitions of n into k nonempty parts.

$P_3(6) = 3$

$6 = 4+1+1 = 3+2+1 = 2+2+2$

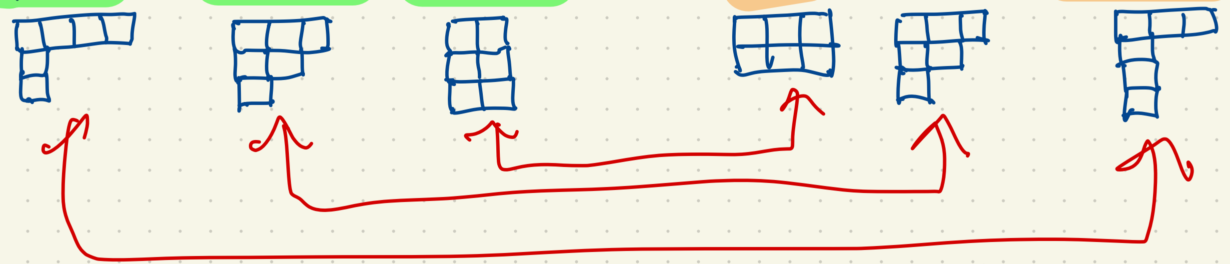
what is the number of partitions of 6 into nonempty parts of maximum size 3 ?

$6 = 3+3 = 3+2+1 = 3+1+1+1$

Theorem $P_k(n)$ = number of partitions of n into nonempty parts of maximum size k.

where $P_k(n)$ is defined as the number of partitions of n into k nonempty parts.

$4+1+1$ $3+2+1$ $2+2+2$ vs. $3+3$ $3+2+1$ $3+1+1+1$



conjugate!
(like transposing matrices: rows $\longleftrightarrow$ columns)

These diagrams are Ferrers diagrams or Young diagrams

The number of partitions of n into parts of size $\le k$ is

$$p_1(n) + p_2(n) + p_3(n) + \cdots + p_k(n)$$

$p_i(n)$ = number of partitions of n into parts of maximum size i

which is also the number of partitions of n into at most k parts.

Let's find generating functions for $p_k(n)$ and $p_1(n)+p_2(n)+\cdots+p_k(n)$.

We'll take k fixed and view this as a sequence indexed by n, namely $p_k(1), p_k(2), p_k(3), \ldots$ where k is fixed

For fixed k, $p_1(n)+p_2(n)+\cdots+p_k(n)$ is the number of partitions of n into parts of size $\le k$ which equals the number of solutions of $n_1+2n_2+\cdots+kn_k = n$ where $n_1, n_2, \ldots, n_k \ge 0$ which is the same as the coefficient of x^n in

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^k+x^{2k}+\cdots) = \sum_{n_1,\ldots,n_k\ge 0} x^{n_1}\cdot x^{2n_2}\cdot\ldots\cdot x^{kn_k} = \sum_{n_1,\ldots,n_k\ge 0} x^{n_1+2n_2+\cdots+kn_k}$$

To get a generating function for $p_k(n)$, take the generating function for $p_1(n)+p_2(n)+\cdots+p_k(n)$ and subtract the generating function for $p_1(n)+p_2(n)+\cdots+p_{k-1}(n)$. This is

$$\frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})(1-x^k)} - \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})(1-x^k)} - \frac{1-x^k}{(1-x)(1-x^2)\cdots(1-x^{k-1})(1-x^k)}$$

$$= \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}$$

Eg. For $k=3$, the generating function for $p_3(n)$ is

$$\frac{x^3}{(1-x)(1-x^2)(1-x^3)} = x^3+x^4+2x^5+3x^6+4x^7+5x^8+7x^9+8x^{10}+\cdots \quad \text{(see Maple session)}$$

$p_3(6)=3$

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$$

$p(n) = p_1(n) + p_2(n) + p_3(n) + \cdots$

$$\sum_{n=0}^{\infty} \left(p_1(n) + p_2(n) + \cdots + p_k(n)\right) x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}$$

The limit of this as $k \to \infty$ is the previous formula.

There is also a linear recurrence formula for $p(n)$ of infinite depth.

The recurrence formula comes from the denominator of the generating function. Recall

$$F_n = \begin{cases} 1, & \text{if } n = 0 \text{ or } 1; \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

gives the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

Its generating function is $\frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + \cdots$

The partition numbers

n	0	1	2	3	4	5	6	7
$p(n)$	1	1	2	3	5	7	11	15

$$\sum_{n=0}^{\infty} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots$$

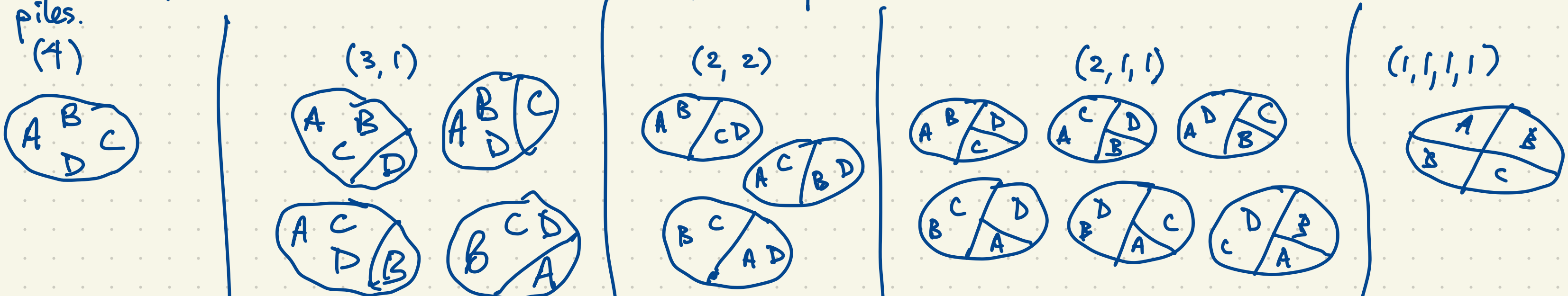
The denominator of $\sum_{n=0}^{\infty} p(n)x^n$ is $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\times\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$

so $p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots = 0$

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$

n	0	1	2	3	4	5	6	7	$\cdots$
$p(n)$	1	1	2	3	5	7	11	15	

Recall: $p(4) = 5$ is the number of ways to partition 4 identical silver dollars into nonempty piles.



The number of ways to divide up 4 (different) students A, B, C, D into nonempty subsets is $B_4 = 15$.

In general the number of ways to partition n students into nonempty subsets is the Bell number B_n. The sequence of Bell numbers is

n	0	1	2	3	4	5	6	7	...
B_n	1	1	2	5	15	52	203	877	...
C_n	1	1	2	5	14	42	...		

Similarity is not a coincidence
$C_n \le B_n$

What is (a) a recurrence formula for B_n?
(b) a generating function for B_n?

Recursively, $B_0 = 1$, $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$

$B_1 = 1B_0 = 1\cdot 1 = 1$

$B_2 = 1B_0 + 1B_1 = 1\cdot 1 + 1\cdot 1 = 2$

$B_3 = 1B_0 + 2B_1 + 1B_2 = 1\cdot 1 + 2\cdot 1 + 1\cdot 2 = 5$

$B_4 = 1B_0 + 3B_1 + 3B_2 + 1B_3 = 1\cdot 1 + 3\cdot 1 + 3\cdot 2 + 1\cdot 5 = 15$

Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

The ordinary generating function of a sequence $a_0, a_1, a_2, a_3, \dots$ is $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

The exponential generating function of $a_0, a_1, a_2, a_3, \dots$ is $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{6} + a_4 \frac{x^4}{24} + \dots$

eg. the sequence $1, 1, 1, 1, 1, 1, \dots$ has ordinary generating function $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$;

its exponential generating function is $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots = e^x$.

Theorem The exponential generating function of B_n is $\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}$.

$F(x) = e^{e^x - 1}$ satisfies $F'(x) = e^{e^x-1} \cdot e^x = F(x) e^x$ and $F(0) = 1$

Proof Write $B(x) = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$ for the exponential generating function of B_n. Then

$$B'(x) = \sum_{n=1}^{\infty} B_n \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} B_{m+1} \frac{x^m}{m!}$$

$\left(\begin{matrix} n = m+1 & \text{i.e.} \\ m = n-1 & \end{matrix} \right)$

$(m,k):\ m \geq k \geq 0$

Set $m = k+l$, $l = m-k$

$$= \sum_{n=0}^{\infty} \left[\sum_{k=0}^{m} \binom{m}{k} B_k \right] \frac{x^m}{m!} = \sum_{m \geq k \geq 0} \frac{1}{m!} \binom{m}{k} B_k x^m = \sum_{m \geq k \geq 0} \frac{1}{\cancel{m!}} \frac{\cancel{m!}}{(m-k)!\, k!} B_k x^m = \sum_{m \geq k \geq 0} \frac{B_k}{k!} \cdot \frac{x^m}{(m-k)!}$$

$$= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{B_k}{k!} \cdot \frac{x^{k+l}}{l!} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} \sum_{l=0}^{\infty} \frac{x^l}{l!} = B(x) e^x$$

$$\frac{B'(x)}{B(x)} = e^x$$

$$\frac{d}{dx} \ln B(x) = e^x$$

$$\ln B(x) = e^x + C$$

$$0 = \ln \underbrace{B_0}_{1} = \ln B(0) = \underbrace{e^0}_{1} + C$$

$$C = -1$$

$$\ln B(x) = e^x - 1$$

$$B(x) = e^{e^x - 1}$$

Go back and prove $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. This is the number of partitions of $[n+1] = \{1, 2, \dots, n, n+1\}$. How many ways can we partition this set? $n+1$ is in a part of the form $A \sqcup \{n+1\}$ where $A \subseteq [n]$.

There are $\binom{n}{k}$ ways to choose this set $A \subseteq [n]$, $|A| = k$. $|A| = k$, $0 \leq k \leq n$.

Then we must partition the remaining $n-k$ elements, which can be done in B_{n-k}. Altogether the number of partitions of $[n+1]$ is

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{l=0}^{n} \binom{n}{n-l} B_l = \sum_{l=0}^{n} \binom{n}{l} B_l .$$

$\begin{cases} l = n-k \\ k = n-l \end{cases}$

$\square$

The number of ways to partition a set of size n into k nonempty parts is the Stirling number $\left\{ {n \atop k} \right\}$. This is the number of ways to partition a pile of n different books into k nonempty piles.

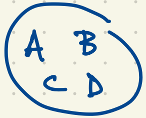
Recall: for n identical silver dollars, $p(n) = p_0(n) + p_1(n) + p_2(n) + \cdots + p_n(n) = \sum_{k=0}^{n} p_k(n)$ ($p_k(n)$ is the number of ways to partition n identical silver dollars into k nonempty piles).

For n different books, $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} = \left\{ {n \atop 0} \right\} + \left\{ {n \atop 1} \right\} + \cdots + \left\{ {n \atop n} \right\}$.

Eg. $n=4$.

$\left\{ {4 \atop 0} \right\} = 0$

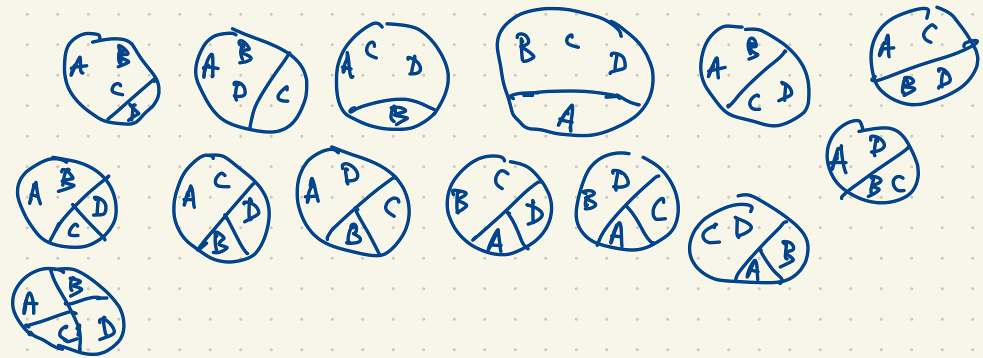
$\left\{ {4 \atop 1} \right\} = 1$



$\left\{ {4 \atop 2} \right\} = 7$

$\left\{ {4 \atop 3} \right\} = 6$

$\left\{ {4 \atop 4} \right\} = 1$



$B_4 = \left\{ {4 \atop 0} \right\} + \left\{ {4 \atop 1} \right\} + \left\{ {4 \atop 2} \right\} + \left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 4} \right\}$

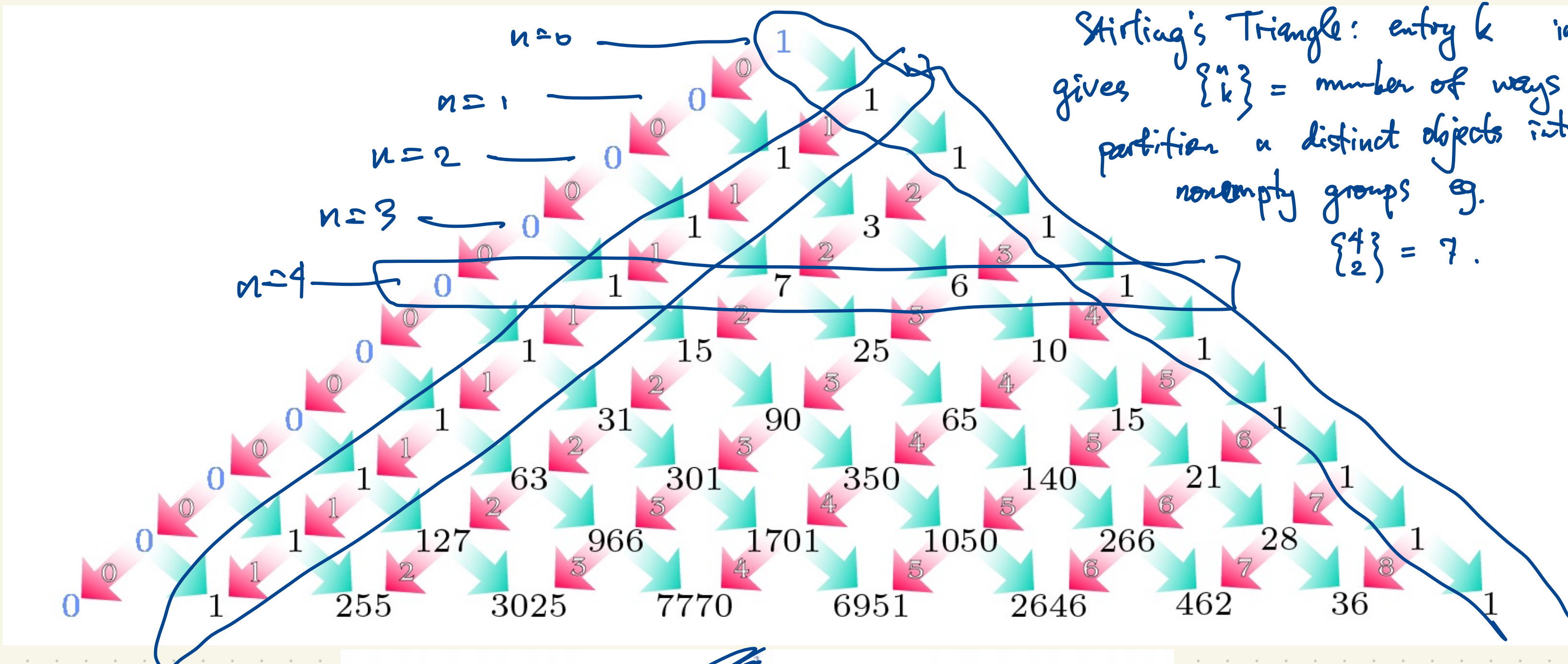
$15 = 0 + 1 + 7 + 6 + 1$

Theorem $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$

Proof The number of ways to partition n students $1, 2, 3, \ldots, n$ into k nonempty groups is

- Partition students $1, 2, \ldots, n-1$ into $k-1$ nonempty groups and add $\{n\}$ as a separate group. $\left\{ {n-1 \atop k-1} \right\}$ ways to do this.
- Partition students $1, 2, \cdots, n-1$ into k nonempty groups and add student n into any of the existing groups in $k \left\{ {n-1 \atop k} \right\}$.

Stirling's Triangle: entry k in row n gives $\left\{ {n \atop k} \right\}$ = number of ways to partition n distinct objects into k nonempty groups eg. $\left\{ {4 \atop 2} \right\} = 7$.



1								
1	1							
1	3	1						
1	7	6	1					
1	15	25	10	1				
1	31	90	65	15	1			
1	63	301	350	140	21	1		
1	127	966	1701	1050	266	28	1	
1	255	3025	7770	6951	2646	462	36	1

$\left\{ {4 \atop 2} \right\} = 7$

Count functions $[k] \to [n]$.

n^k functions altogether ($f(1)$ can be chosen in n ways
$f(2)$ - - — — - .
$\vdots$
$f(k)$ - - - . - . .) $\qquad \underbrace{n \times n \times \cdots \times n}_{k} = n^k$.

How many $\underbrace{\text{one-to-one (injective) functions}}_{\text{injections}}$ $[k] \to [n]$?

$f(1)$: n choices
$f(2)$: $n-1$ ..
$f(3)$: $n-2$ ...
$\vdots$
$f(k)$: $n-k+1$ ..

$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$
or $(n)_k$ or $[n]_k$ or $n_{(k)}$

How many functions from $[k]$ onto an n-set $[n]$?
onto = surjective
Functions that are onto are surjections.

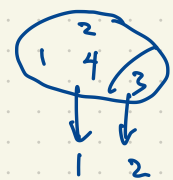
First partition $[k]$ into n nonempty subsets in $\left\{ {k \atop n} \right\}$ then match up these subsets to the values $1, 2, \dots, n$ in $n!$ ways

Altogether $\left\{ {k \atop n} \right\} n!$ surjections $[k] \to [n]$

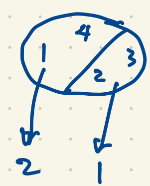
(or $\left\{ {n \atop k} \right\} k!$ surjections $[n] \to [k]$).

Eg. a surjection from $[4] = \{1,2,3,4\}$ to $[2] = \{1,2\}$:

x	$f(x)$
1	1
2	1
3	2
4	1



x	$g(x)$
1	2
2	1
3	1
4	2



Eg. How many ways can I distribute 7 identical silver dollars to 3 students?
what if we need every student to receive at least one of the silver dollars?

$\binom{7+3-1}{3-1} = \binom{9}{2} = 36$ ways to distribute 7 silver dollars to 3 students.

(Bitstrings of length 9 with 7 zeroes and 2 ones correspond to solutions e.g.

$\underbrace{00}1\underbrace{0}1\underbrace{0000}$: 2 silver dollars to student 1

1 — — — — 2

4 — — — — 3)

If each student should receive at least one silver dollar, first distribute one silver dollar to each student, then $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$ ways.

Eg. How many ways can I distribute 7 different books to 3 students? What if every student should receive at least one book?

$3^7 = 2187$ ways in all.

If each student should receive at least one book, $\left\{ {7 \atop 3} \right\} 3! = 301 \cdot 6 = 1806$ ways.

General formulas:

$$\left\{ {n \atop k} \right\} = 1 \quad \text{if } k=1,\ n \geq 1$$

$$\left\{ {n \atop n} \right\} = 1$$

$$\left\{ {n \atop 0} \right\} = \begin{cases} 0 & \text{if } n \geq 1; \\ 1 & \text{if } n = 0 \end{cases}$$