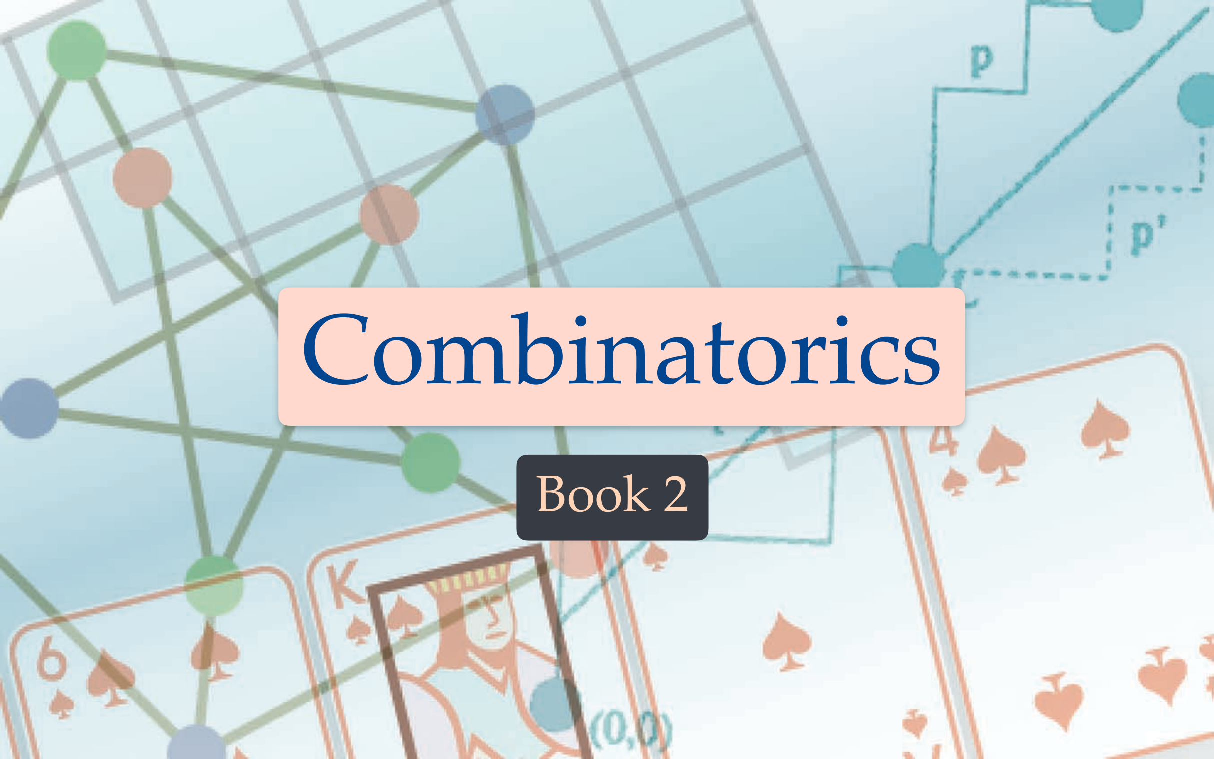
Combinatorics

Book 2



The graph $\Gamma \setminus v$ (formed by removing v and its edges from Γ) has one fewer vertex, so it can be properly colored using at most 6 colors. And since v has at most 5 neighbors in $\Gamma \setminus v$, there is a color left over which can be used to color vertex v. This gives a proper coloring of Γ using at most 6 colors (a contradiction ...)

We will improve this to show that actually 5 colors suffice to properly color every planar graph.

$\psi \ \phi \ \chi \ \pi$

A b c ... x X χ

Greek "chi"

Given a graph Γ, the <u>chromatic</u> <u>number</u> of Γ, denoted $\chi(\Gamma)$, is the smallest number of colors we can use to properly color the vertices of Γ. A <u>proper</u> <u>coloring</u> of the vertices of Γ is a coloring of the vertices such that no edge has both endpoints of the same color.

The theorem of Appel and Haken (1976) is that every planar graph Γ has $\chi(\Gamma) \leq 4$.

Note that $\chi(K_n) = n$. Here K_n is the complete graph of order n.

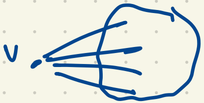
A graph Γ has $\chi(\Gamma) = 1$ iff it has vertices but no edges.

A graph Γ has $\chi(\Gamma) \leq 2$ iff Γ is bipartite iff Γ has no circuits of odd length.

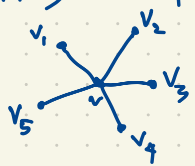
<u>Computing</u> $\chi(\Gamma)$ is hard in general.

<u>Theorem</u> If Γ is a finite planar graph then $\chi(\Gamma) \leq 5$. Proof due to Heawood.

<u>Proof</u> If the theorem fails then there is a smallest counterexample Γ with n vertices (so Γ is planar and every planar graph of order $n-1$ has chromatic number ≤ 5 while $\chi(\Gamma) \geq 6$). We seek a contradiction. Γ has a vertex v of degree ≤ 5. In fact $\deg v = 5$. (If $\deg v \leq 4$ then $\chi(\Gamma) \leq 5$, a contradiction.) Let Γ' be the graph obtained



$n-1$ vertices can be properly colored using at most five colors.

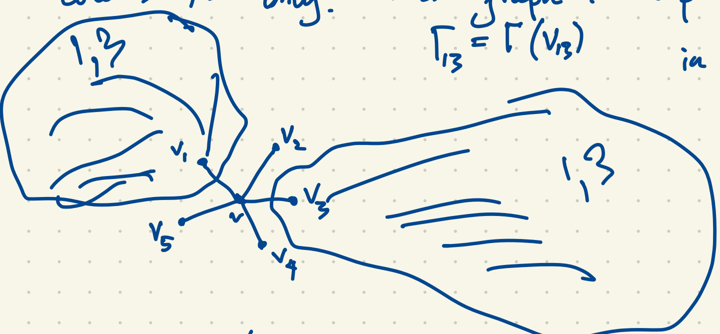


from Γ by deleting v and its five edges, so $\chi(\Gamma') \leq 5$. Say v_i has color i $(i = 1, 2, \cdots, 5)$.

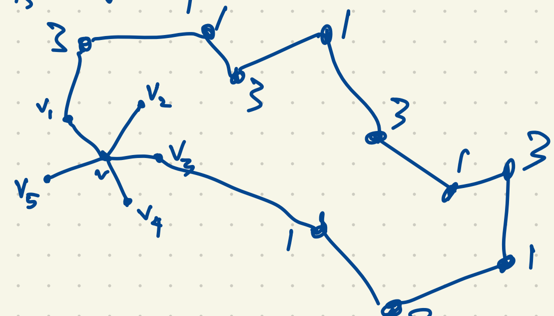
Consider the vertices $V_{13} \subset \{\text{vertices of } \Gamma\}$ having colors 1, 3 only. This graph is bipartite. I can assume v_1 is joined to v_3 in Γ_{13} (otherwise

$\Gamma_{13} = \Gamma(V_{13})$

in part of Γ_{13}, reverse colors 1, 3 so that v_3 gets color 1. Then we are free to color v using color 3 since its neighbors are color 1, 2, 1, 4, 5).

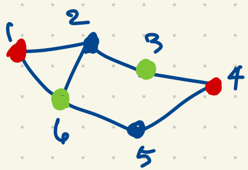


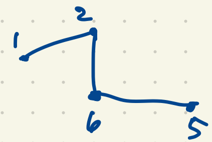
Otherwise Γ_{13} has a path from v_1 to v_3.

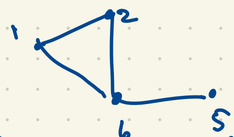


Similarly there is a path from v_2 to v_4 using only vertices of colors 2 and 4. Contradiction! $\square$

Given a graph Γ, a <u>subgraph</u> of Γ is formed by taking a subset of the edges of Γ together with all their vertices. An <u>induced subgraph</u> of Γ is formed by taking a subset of the vertices of Γ together with all their edges in Γ

$\Gamma =$ 

eg.  is a subgraph of Γ. (not an induced subgraph of Γ)

eg.  is an induced subgraph of Γ.

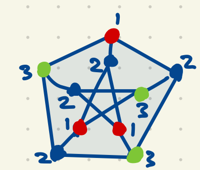
An induced subgraph of Γ is a subgraph of Γ, but not conversely.

A <u>k-clique</u> in Γ is a complete subgraph of Γ, i.e. a subset of the vertices, any two of which are joined.

In Γ above, $\{1,2,6\}$ is a clique (in fact a 3-clique). The <u>clique number</u> of Γ, denoted $w(\Gamma)$, is the size of the largest clique in Γ. It is hard to compute $w(\Gamma)$.

W vs. ω
Roman Greek

<u>Theorem</u> For every graph Γ, $\chi(\Gamma) \geq \omega(K)$.



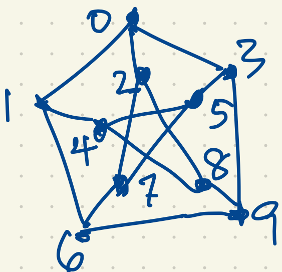
$\chi(P) = 3$ for the Petersen graph P.

Warning: this not equality! For the Petersen graph P, $w(P) = 2$.

Proof: The vertices in a clique of size $w(\Gamma)$ require $w(\Gamma)$ different colors.

Dual to the clique number $\omega(\Gamma)$ we have the coclique number $\alpha(\Gamma)$ which is the maximum number of vertices in Γ, no two of which are joined. (This is $\alpha(\Gamma) = \omega(\bar{\Gamma})$ where $\bar{\Gamma}$ is the complementary graph). Cocliques are also called independent sets of vertices.

Eg. $\alpha(P) = 4$. $\quad |\chi(P)| \geq \frac{10}{4} = 2.5 \implies \chi(P) \geq 3$.



$\{1,2,3\}$ is a coclique which is not contained in any larger coclique; it is a <u>maximal</u> coclique.

A <u>maximum</u> coclique (i.e. a coclique of maximum size) is $\{1,3,7,8\}$.

This is maximum size because P has vertex set $\{0,3,9,6,1\} \cup \{2,7,5,4,8\}$ as a union of two 5-cycles (circuits of length 5). Any set ~~of~~ size at least 5 vertices has either 3 on the inner 5-cycle $\{2,7,5,4,8\}$ or 3 vertices on the outer 5-cycle $\{0,3,9,6,1\}$. In either case there is an edge in that 5-cycle joining two vertices we have chosen.

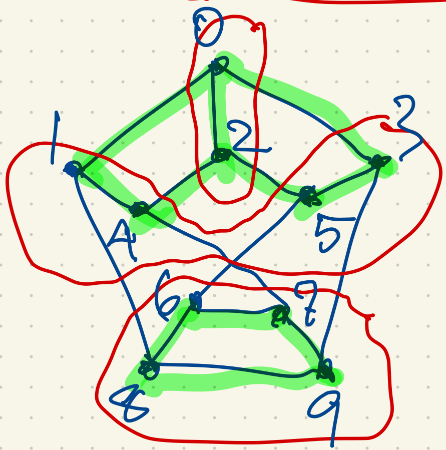
<u>Theorem</u> $\chi(\Gamma) \geq \dfrac{|V|}{\alpha(\Gamma)}$ where $|V|$ = the number of vertices = the order of Γ.

<u>Proof</u> Let $k = \chi(\Gamma)$. Properly color the vertices $1, 2, \cdots, k$ and let V_i be the subset of vertices colored i, for $i = 1, 2, \cdots, k$. This gives a partition $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ ($V_i \cup V_j$ = union of V_i and V_j; $V_i \sqcup V_j$ = disjoint union of V_i and V_j). Each V_i is a coclique so $|V_i| \leq \alpha(\Gamma)$ so $|V| = |V_1| + |V_2| + \cdots + |V_k| \leq \underbrace{\alpha(\Gamma) + \alpha(\Gamma) + \cdots + \alpha(\Gamma)}_{k} = k\,\alpha(\Gamma)$ so $\chi(\Gamma) = k \geq \dfrac{|V|}{\alpha(\Gamma)}$. $\square$

March	M	W	F
		1	3
	6	$\boxed{8}$	10
	13	15	17

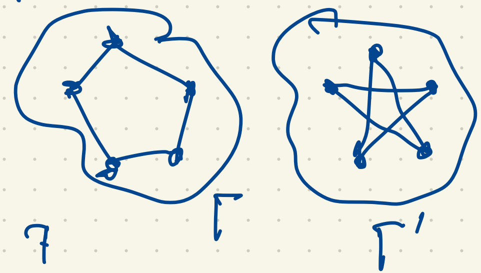
Spring Break

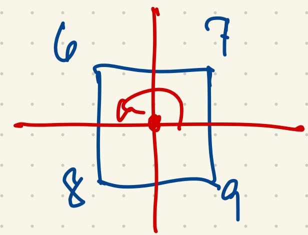
Test 1: Wed Mar 8



$= \Gamma$

You can use nauty to test isomorphism between two graphs.





Using nauty Software

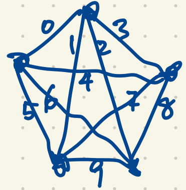
$G = \operatorname{Aut} \Gamma$

$G = \langle (1,3)(4,5)(6,7)(8,9),\ (0,2)(1,4)(3,5)(6,9)(7,8) \rangle$

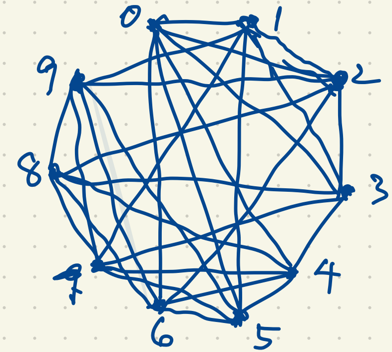
$|G| = 4$

G has 3 orbits on the vertices: $\{0,2\}, \{1,3,4,5\}, \{6,7,8,9\}$

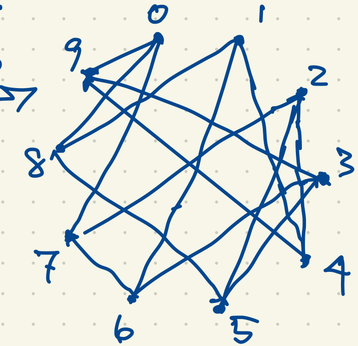
$K_5 =$



K_5'

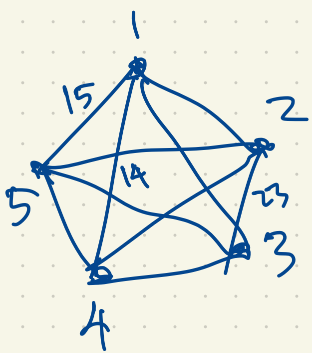


$\xrightarrow{\overline{K_5'}}$

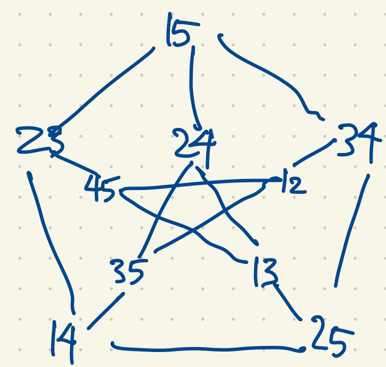


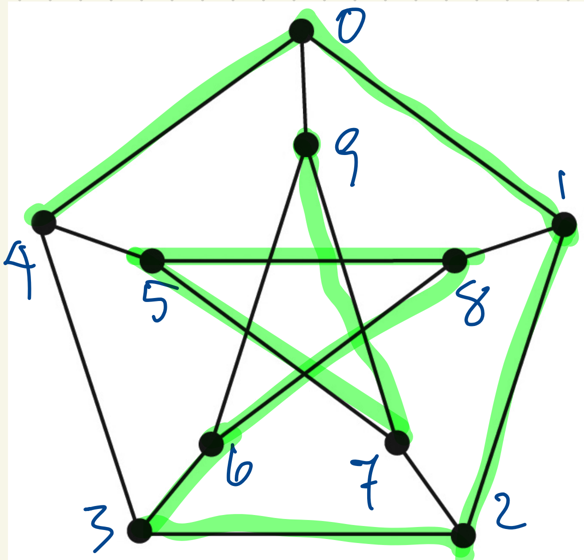
$|\mathrm{Aut}\ \overline{K_5'}| = 120$

$K_5 =$

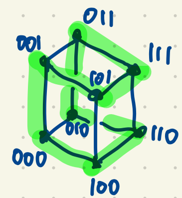


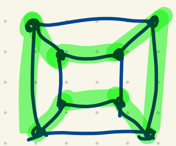
$\overline{K_5'} =$





The Petersen graph P has a Hamilton path $(0,1,2,3,6,8,5,7,9)$ (a path touching each vertex exactly once) but no Hamilton circuit (ending at the same vertex where it started).

The Hamming cube H_3 = 

= 

does have a Hamilton circuit.

000
100
110
010
011
111
101
001
000

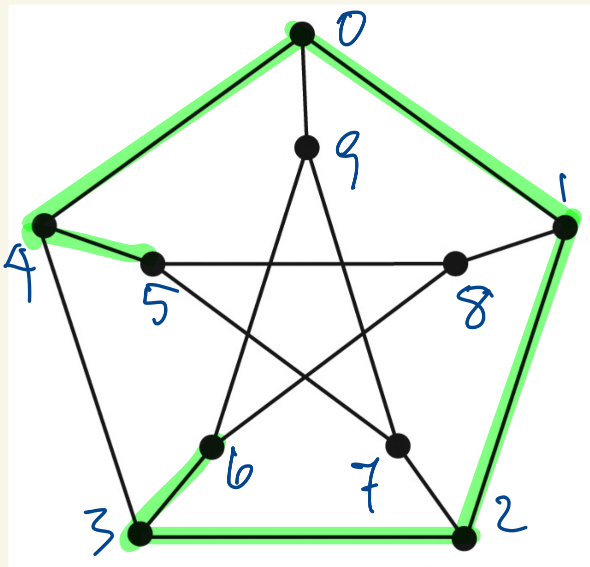
"Gray code"

Every Hamming graph H_n $(n \geqslant 2)$ has a Hamilton circuit.

A graph having a Hamilton circuit is called <u>Hamiltonian</u>.

Looking for Hamilton paths or circuits is known to be difficult in general.

Testing whether a giving graph Γ is Hamiltonian is NP-complete.

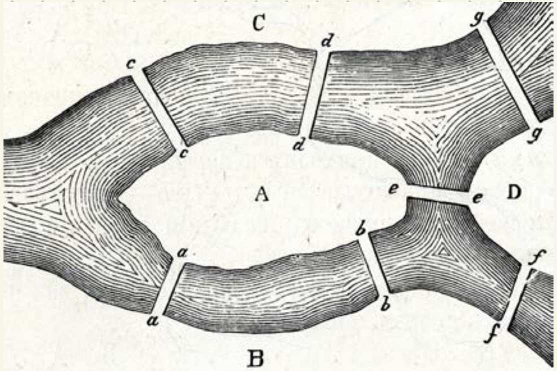


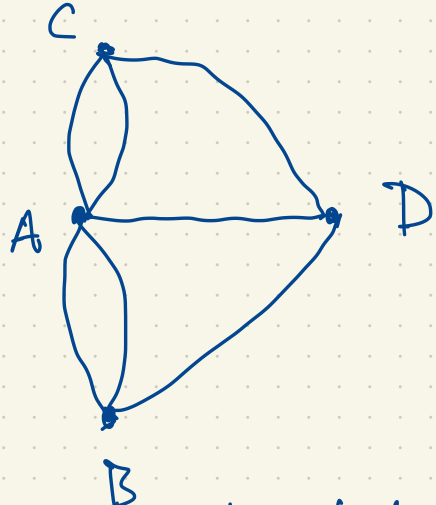
<u>Theorem</u>: The Petersen graph P is not Hamiltonian, i.e. it does not have a Hamilton circuit/cycle.

<u>Proof</u> Suppose P has a Hamilton circuit. Without loss of generality this circuit contains the path (4, 0, 1, 2). (This is because P has 120 automorphisms mapping any such path of length 3 to any other.) The Hamilton circuit uses two of the edges from vertex 3, so it uses either {3,4} or {2,3}; so without loss of generality, it uses the edge {2,3}. We cannot use the edge {3,4} as this would complete the circuit without passing through all vertices; so we must use the edges {3,6} and {4,5}. To continue the circuit from vertex 6, we have two choices: proceed through vertex 8 or vertex 9. Neither of these choices leads to a Hamilton circuit. This is a contradiction. □

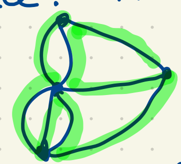
Euler paths and circuits

The Seven bridges of Königsberg





An <u>Euler</u> <u>trail</u> is a trail (repeating vertices but not edges) which uses each edge exactly once. An <u>Euler circuit</u> is an Euler trail that returns to its starting point.



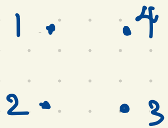
This graph has an Euler trail. In order to have an Euler trail, a graph must have either 0 or 2 vertices of odd degree. When there are no vertices of odd degree, we have an Euler circuit.

<u>Theorem</u> (Euler) A graph has an Euler trail iff it is connected and it has either 0 or 2 vertices of odd degree. In the case every vertex has even degree, we have an Euler circuit/cycle.

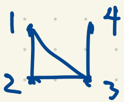
We sometimes speak of labelled graphs and unlabelled graphs.

Eg. on the vertex set $\{1,2,3,4\}$, there are $2^6 = 64$ labelled graphs

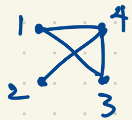
$\binom{4}{2} = 6$ pairs of vertices.



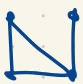
There are $\binom{n}{2}$ labelled graphs on n vertices.
But many of them are isomorphic.



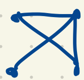
$\neq$



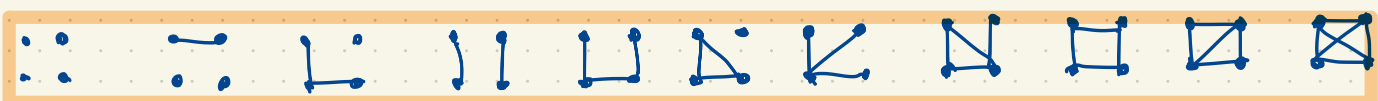
These are different labelled graphs but they are isomorphic.



$\stackrel{?}{=}$



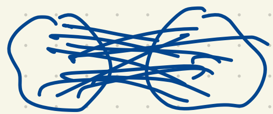
As unlabelled graphs they are isomorphic, hence the same graph.



There are 11 unlabelled graphs of order 4 i.e. 11 isomorphism types of graphs of order 4, i.e. 11 graphs of order 4 up to isomorphism.

The Petersen graph has girth 5 (the shortest cycle has length 5). It has 15 edges.
For a graph on 10 vertices, 15 edges is the maximum possible for girth 5.
For a graph on 10 vertices without triangles (i.e. girth ≥ 4), what is the maximum possible number of edges?

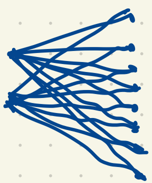
Recall: $K_{m,n}$ =



m n

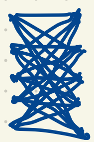
$K_{m,n}$ has mn edges. $K_{m,n}$ is bipartite so it has no cycles of odd length. In particular it has no triangles.

$K_{2,8}$ has 16 edges



$K_{2,8}$

16 edges



$K_{5,5}$

25 edges
girth 4
(no triangles)

<u>Theorem</u> (Mantel 1907)
If Γ is a graph of order n with no triangles (i.e. its girth is at least 4) then Γ has at most $\frac{n^2}{4}$ edges.

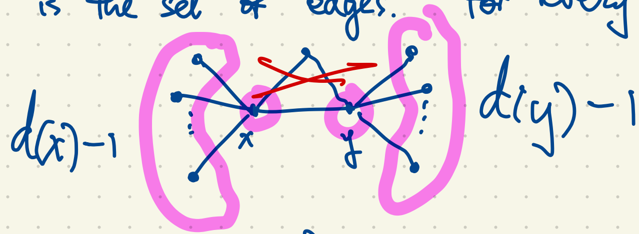
If n is even then $K_{\frac{n}{2},\frac{n}{2}}$ attains the upper bound of $\frac{n^2}{4}$ edges. What if n is odd?

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even, for } K_{\frac{n}{2},\frac{n}{2}} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd, for } K_{\frac{n+1}{2},\frac{n-1}{2}} \end{cases}$$

On 9 vertices, any graph without triangles has at most <u>20</u> edges.

$K_{4,5}$

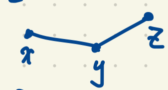
Proof Let Γ be a graph of order n with no triangles, $\Gamma = (V, E)$. (V is the set of vertices, E is the set of edges.) For every edge $\{x,y\} \in E$, $d(x) + d(y) \le n$.



$$d(x) - \cancel{1} + \cancel{1} + \cancel{1} + d(y) - \cancel{1} \le n$$

Add the inequality $d(x) + d(y) \le n$ over all edges $\{x,y\} \in E$ to get $\sum_{\{x,y\} \in E} (d(x) + d(y)) \le ne$.

Next, count the number of triples of vertices (x, y, z) with $x \sim y \sim z$.

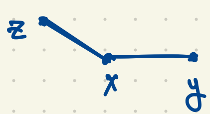


There are n choices for $y \in V$ and $d(y)$ choices for x, $d(y)$ choices for z, so $d(y)^2$ choices for x and z (given y). The total number of walks of length 2 is $\sum_{y \in V} d(y)^2$.

On the other hand, there are $e = |E|$ edges in Γ. For the edge $\{x,y\} \in E$, how many walks of length 2 contain this edge? $d(x) + d(y)$ choices of walk of length 2 in which we include a step from x to y

The total number of walks of length 2 is

$$\sum_{\{x,y\} \in E} (d(x) + d(y)).$$



$d(x)$ choices for z (given the edge $\{x,y\}$)

or

$d(y)$ choices for z (given the edge $\{x,y\}$)

Therefore

$$\sum_{x\in V} d(x)^2 = \sum_{\{x,y\}\in E} \underbrace{(d(x)+d(y))}_{\le n} \le en$$

If $\sum_{x\in V} d(x) = 2e$, what does this tell us about $\sum_{x\in V} d(x)^2$?

Use the Cauchy-Schwarz inequality.

$$(\underbrace{d(1)+\cdots+d(n)}_{2e})^2 \le n(d(1)^2+\cdots+d(n)^2)$$

$$4e^2 \le n\sum_{x\in V} d(x)^2 = n\sum_{\{x,y\}\in E}(d(x)+d(y)) \le n\cdot ne = n^2e$$

So $e \le \frac{n^2}{4}$.

Cauchy-Schwarz Inequality

Given two vectors

$a = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$

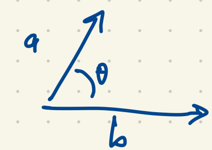
$b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$,

$$|a\cdot b| \le \|a\|\cdot\|b\|$$

where $a\cdot b = a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$$a\cdot b = \|a\|\,\|b\|\cos\theta$$



$|\cos\theta| \le 1$.

Special case: $b = (1,1,1,\cdots,1)$

$\|b\| = \sqrt{\underbrace{1^2+1^2+\cdots+1^2}_{n}} = \sqrt{n}$

$a\cdot b = a_1 + a_2 + \cdots + a_n$

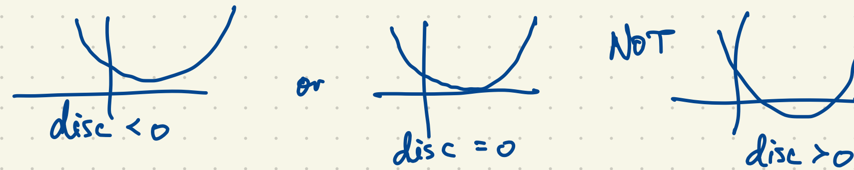
$|a\cdot b| \le \|a\|\cdot\sqrt{n}$

$$(a_1+\cdots+a_n)^2 \le n(a_1^2+\cdots+a_n^2)$$

Proof of Cauchy-Schwarz: Fix $a, b \in \mathbb{R}^n$. Consider

$$f(t) = \|a - tb\|^2 = (a-tb)\cdot(a-tb) = a\cdot a - tb\cdot a - ta\cdot b + t^2\, b\cdot b = \|a\|^2 - 2t(a\cdot b) + t^2\|b\|^2.$$

Of course $f(t) \geq 0$ for all t.



The discriminant

$$(-2a\cdot b)^2 - 4\|a\|^2\|b\|^2 \leq 0$$

$$4(a\cdot b)^2 \leq 4\|a\|^2\|b\|^2$$

$$(a\cdot b)^2 \leq \|a\|^2\|b\|^2.$$

The disc. of at^2+bt+c is b^2-4ac