University of Wyoming

Math 3700—Spring 2023

Combinatorics

Department of Mathematics

Sample Test 2

April, 2023

This sample test is intended to resemble the Optional Second Test in approximate length, difficulty, and style, although clearly the content may differ. The actual content will emphasize material covered in class since the first test, particularly further use of binomial and multinomial coefficients, Catalan numbers, counting strings or words over a given alphabet (including bitstrings over a binary alphabet $\{0,1\}$), and generating functions.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an $8.5''\times 11''$sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 50 minutes.

1. For every positive integer n, denote by a_n the number of bitstrings of length $2n$ having n zeroes and n ones.
   (a) Express the generating function $\sum_{n=0}^{\infty} a_n x^n$ in simple closed form.
   (b) Tabulate a_n for $n \in \{0,1,2,3,4,5\}$ and verify that a_n is divisible by $n{+}1$ in each case.
   (c) Is a_n divisible by $n{+}1$ for every $n \geqslant 0$? *Justify your answer.*

2. Recall that the standard n-set is $[n] = \{1,2,\ldots,n\}$. State
   (a) the number of functions $[5] \to [5]$;
   (b) the number of surjective functions $[5] \to [3]$ (i.e. the number of functions from $[5]$ *onto* $[3]$);
   (c) the number of injective functions $[3] \to [5]$ (i.e. the number of one-to-one functions from $[3]$ to $[5]$);
   (d) the number of functions $[5] \to [3]$.

3. Let $f(x,y,z,w) = (x+2y+3z+4w)^{10}$.
   (a) How many terms are in the expansion of $f(x,y,z,w)$?
   (b) What is the coefficient of $x^3yz^2w^4$ in $f(x,y,z,w)$?
   (c) What is the sum of the coefficients in $f(x,y,z,w)$?

4. (a) How many ways can I give eleven identical coins to eight students?
   (b) How many ways can I give eleven identical coins to eight students, if each student is expected to receive at least one of the coins?

(c) How many ways can I give eleven different coins to eights students?

5. Define a sequence $a_0, a_1, a_2, \ldots$ recursively by $a_0 = 1$; and for all $n \geqslant 1$, a_n is the sum of *all* the previous terms in the sequence. Determine
   (a) a closed formula for a_n, and
   (b) an explicit formula for the generating function of the sequence.

6. Answer TRUE or FALSE to each of the following statements.
   (a) The number of surjections $[n] \to [k]$ equals the number of injections $[k] \to [n]$, by simply reversing all the arrows. ______*(True/False)*
   (b) Every power series $\sum_{n=0}^{\infty} a_n x^n$ is expressible as a rational function of x. ______*(True/False)*
   (c) Every rational function of x is expressible as a power series $\sum_{n=0}^{\infty} a_n x^n$. ______*(True/False)*
   (d) The number of 11-free ternary strings of length n grows at an exponential rate as $n \to \infty$. (A string over the ternary alphabet $\{0, 1, 2\}$ is called ternary.) ______*(True/False)*
   (e) The number of subsets of an n-set forms a sequence whose generating function is $\frac{1}{1-2x}$. ______*(True/False)*
   (f) If m is any real number, then the binomial expansion of $(1+x)^m$ is a power series, all of whose coefficients are non-negative integers. ______*(True/False)*
   (g) If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$, then the generating function for the sequence $a_0 b_0,\ a_1 b_1,\ a_2 b_2,\ \ldots$ is $A(x)B(x)$. ______*(True/False)*
   (h) If u, v are two vertices in a finite graph Γ, and a_n is the number of walks of length n from vertex u to vertex v in Γ, then the generating function for a_n is a rational function of x. ______*(True/False)*
   (i) The generating function $F(x) = \sum_{n=0}^{\infty} F_n x^n$ for the Fibonacci sequence $F_0, F_1, \ldots$ satisfies $F(x) = F(x-1) + F(x-2)$. ______*(True/False)*
   (j) If a_n is the number of bitstrings of length n that are both 00-free and 11-free, then the generating function for a_n is $\frac{1+x}{1-x}$. (A string is 00-free if it dos not contain two consecutive 0's.) ______*(True/False)*