Sample Test 1

March, 2023

Test 1 will take place on Wednesday, March 8, 2023 during class time. This sample test is intended to resemble Test 1 in approximate length, difficulty, and style, although clearly the content may differ. The actual content will be selected from all class lectures prior to the test.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 17 bonus points).

All graphs here are undirected, with no loops or multiple edges.

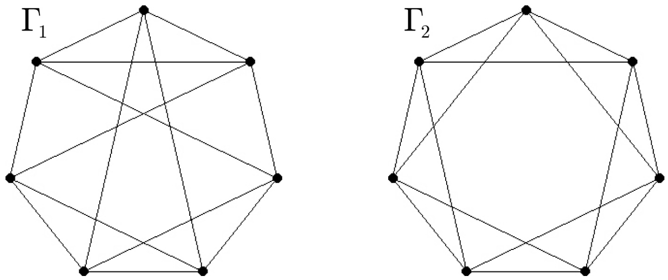
1. *(18 points)* A graph Γ of order 5 has vertex set {bat, got, peg, put, rag}. Two vertices are joined if their names have one letter in common (so 'rag' is joined to 'got', but not to 'put').
   (a) Draw the graph Γ as simply as possible.
   (b) Is Γ planar?
   (c) Determine the clique number $\omega(\Gamma)$.
   (d) Determine the coclique number (i.e. independence number) $\alpha(\Gamma)$.
   (e) Determine the chromatic number $\chi(\Gamma)$.
   (f) How many automorphisms does Γ have?

2. *(10 points)* Give an example of a planar graph Γ (having more than one vertex and more than one edge) whose dual planar graph is isomorphic to the original graph Γ.

3. *(15 points)* Let n, k be integers with $1 \leqslant k \leqslant n$, and let $[n] = \{1, 2, \ldots, n\}$. The *Kneser graph* $KG_{n,k}$ has as its vertices the k-subsets of $[n]$, i.e. the subsets $A \subseteq [n]$ such that $|A| = k$. Two vertices A, B are adjacent iff they are disjoint (i.e. $A \sim B$ iff $A \cap B = \varnothing$). Draw (a) $KG_{3,2}$, (b) $KG_{4,2}$, and (c) $KG_{5,2}$.

4. *(12 points)* Let Γ be a graph, and let $\overline{\Gamma}$ be its complementary graph.
   (a) If Γ is disconnected, what is the largest possible value for the diameter of $\overline{\Gamma}$?
   (b) Is it possible for a graph and its complement to both be disconnected? Explain.

5. *(20 points)* Consider the following two graphs:



(a) Determine the clique number $\omega(\Gamma_i)$, coclique number $\alpha(\Gamma_i)$ and chromatic number $\chi(\Gamma_i)$ in each case:

$\omega(\Gamma_1) =$ ___ $\alpha(\Gamma_1) =$ ___ $\chi(\Gamma_1) =$ ___

$\omega(\Gamma_2) =$ ___ $\alpha(\Gamma_2) =$ ___ $\chi(\Gamma_2) =$ ___

(b) Are the graphs Γ_1 and Γ_2 isomorphic? Explain.

(c) Determine the number of automorphisms of each of the graphs Γ_i.

$|\operatorname{Aut}\Gamma_1| =$ ___ $|\operatorname{Aut}\Gamma_2| =$ ___

6. *(12 points)* There are $2^{\binom{10}{2}} = 2^{45}$ labelled graphs on 10 vertices, i.e. there are 2^{45} different ways to put edges on the vertex set $[10] = \{1, 2, \ldots, 10\}$. How many of these labelled graphs are isomorphic to the Petersen graph? (Your answer should be expressed as an explicit five-digit decimal number.)

7. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) For all $n \geqslant 2$, the Hamming n-cube H_n has an Euler circuit. ______*(True/False)*

(b) For all $m, n \geqslant 2$, the complete bipartite graph $K_{m,n}$ has a Hamilton circuit. ______*(True/False)*

(c) There exists a graph Γ of order 5 which is isomorphic to its complement. ______*(True/False)*

(d) There exists a graph with degree sequence $(1, 1, 2, 2, 3, 3)$. ______*(True/False)*

(e) There exists an infinite connected graph having infinite diameter. ______*(True/False)*

(f) A finite 2-regular graph is necessarily a disjoint union of cycles.______*(True/False)*

(g) A connected graph of order n must have at least $n-1$ edges. ______*(True/False)*

(h) The complete graph K_n has $n!$ automorphisms. ______*(True/False)*

(i) If every vertex in a graph Γ has degree at most 3, then $\chi(\Gamma) \leqslant 4$. ______*(True/False)*

(j) If Γ is a 3-regular finite graph, then its number of edges must be a multiple of 3. ______*(True/False)*