University of Wyoming

Math 3700—Spring 2023

# Combinatorics

Department of Mathematics

$C(x) = \sum_{n=0} \binom{2n}{n} \frac{x^n}{n+1}$

## Solutions to Practice Problems

More complete solutions will be posted at a later date (the problems are not hard, but I probably need at least another 50 minutes to type up the solutions).

1. (a) $\binom{11+8-1}{11} = \binom{18}{7} = 31{,}824$. This is the same as counting bitstrings of length 18 having eleven 1's (representing the coins) and seven 0's (as separators between students).

   (b) $\binom{3+8-1}{3} = \binom{10}{3} = 120$. First give one coin to each student, then hand out the remaining 3 coins. For this, count bitstrings of length 10 having three 1's (for the three remaining coins) and seven 0's (as separators between the students).

   (c) $8^{11} = 8{,}589{,}934{,}592$. Here we count all functions from an 11-set (the set of coins) to an 8-set (the set of students).

   (d) As in (c), except this time we only count surjections. There are $\left\{ {11 \atop 8} \right\} 8! = 479{,}001{,}600$ functions from the 11-set of coins *onto* the 8-set of students.

2. Starting with $G(x) = (1-x)^{-1} = \sum_{n=0}^{\infty}$, we obtain $G'(x) = (1-x)^{-2} = \sum_{n=0}^{\infty}(n+1)x^n$ and $G''(x) = 2(1-x)^{-3} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n$. It remains only to find the right linear combination (by solving three equations in three unknowns): $S(x) = G''(x) - 3G'(x) + G(x) = 2(1-x)^{-3} - 3(1-x)^{-2} + (1-x)^{-1} = (1-x)^{-3}[2 - 3(1-x) + (1-x)^2] = \frac{x+x^2}{(1-x)^3}$.

3. (a) From $f(x) = (1+x)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^k$ we obtain $f'(x) = n(1+x)^{n-1} = \sum\limits_{k=0}^{n} \binom{n}{k} k x^{k-1}$. So $f'(1) = \sum\limits_{k=0}^{n} \binom{n}{k} k = 2^{n-1} n!$.

   (b) Continuing from (a), $f''(x) = n(n-1)(1+x)^{n-2} = \sum\limits_{k=0}^{n} \binom{n}{k} k(k-1)x^{k-2}$, so $f''(1) = 2^{n-2}n(n-1) = \sum\limits_{k=0}^{n} \binom{n}{k} k(k-1)$. Adding to the value in (a) gives $f'(1) + f''(1) = \sum\limits_{k=0}^{n} \binom{n}{2} k^2 = 2^{n-2}n(n+1)$.

   (c) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

   *First Proof*. Consider a $2n$-set $A \sqcup B$ where $A$ and $B$ are disjoint sets of size $n$. (Here 'disjoint' means that $A \cap B = \varnothing$; and in this case it is customary to write their union $A \cup B$ as $A \sqcup B$ to emphasize that this is a 'disjoint union'.) Every $n$-subset of $A \sqcup B$ has the form $X \sqcup Y$ for some $k$-subset $X \subseteq A$ and some $(n-k)$-subset $Y \subseteq B$, so the number of $n$-subsets of $A \sqcup B$ is $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$.

*Second Proof*. Since $(1+x)^{2n} = (1+x)^n(1+x)^n = \left[\sum_{k=0}^n \binom{n}{k}x^k\right]\left[\sum_{k=0}^n \binom{n}{k}x^k\right]$, we expand the right hand side as a convolution and compare the coefficient of $x^n$ on both sides to obtain the required identity.

4. (a) $\binom{17}{5} = 6188$. Each path is encoded as a string of length 17 having twelve E's and five N's (interpreted as instructions for when to walk east, and when to walk north).

   (b) Before proceeding with our solution, let's give an example generalizing the computation of (a): we have

$$\begin{aligned} F_{12,5}(x) = {} & 6188x^{17} + 515508x^{19} + 24418800x^{21} + 868346094x^{23} + 25843633750x^{25} \\ & + 681061288050x^{27} + 16434404695800x^{29} + 371087004526875x^{31} \\ & + 7957932266924640x^{33} + 163806440537124000x^{35} + \cdots \end{aligned}$$

The coefficient of $x^n$ in this generating function is $w_{i,j}(n)$, the number of walks in the grid of length $n$ blocks from $(0,0)$ to $(i,j)$. Of course every such walk has length $n \geqslant 17$. Moreover every walk from $(0,0)$ to $(12,5)$ has odd length; that is because the grid gives a bipartite graph. (We may color the vertices $(i,j)$ either red or blue according as $i+j$ is even or odd. Every block in the grid joins a red vertex and a blue vertex. For more about this concept, refer to Practice Problems 2.) It is fairly clear that the coefficients in our generating function grow at a superpolynomial rate. As we shall see, it is possible to express the coefficient $w_{i,j}(n)$ as a sum of multinomial coefficients; but as we shall see, it is much more convenient to read off coefficients of $E^iN^j$ in a certain expansion obtained from walk generating functions.

We represent the intersection $(i,j)$ as the monomial $E^iN^j$ using two indeterminates $E$ and $N$, since this is $i$ blocks east and $j$ blocks north of the main intersection. In order to allow arbitrary integer coordinates (positive, negative and zero), rather than working in the polynomial ring $\mathbb{Z}[E,N]$, we use $R = \mathbb{Z}[E,E^{-1},N,N^{-1}]$, the ring of all polynomials in two indeterminates $E$ and $N$ with arbitrary integer exponents. (In other words, $R$ is the set of all finite sums of the form $\sum_{i,j} c_{i,j}E^iN^j$ where the coefficients $c_{i,j} \in \mathbb{Z}$. Here $i,j \in \mathbb{Z}$; and for elements of $R$, we allow only finitely many nonzero terms of this form.) The set of all possible walks of length 1 starting at the origin is represented by $A = E + E^{-1} + N + N^{-1} \in R$. Now $A^n = \sum_{i,j\in\mathbb{Z}} w_n(i,j)E^iN^j \in R$ represents all the walks of length $n$ starting at the origin, where the sum extends over all integers $i,j$ (but for each $n$, there are only finitely many nonzero terms since the coefficient $w_n(i,j)$ is zero when $|i|+|j| > n$). Next,

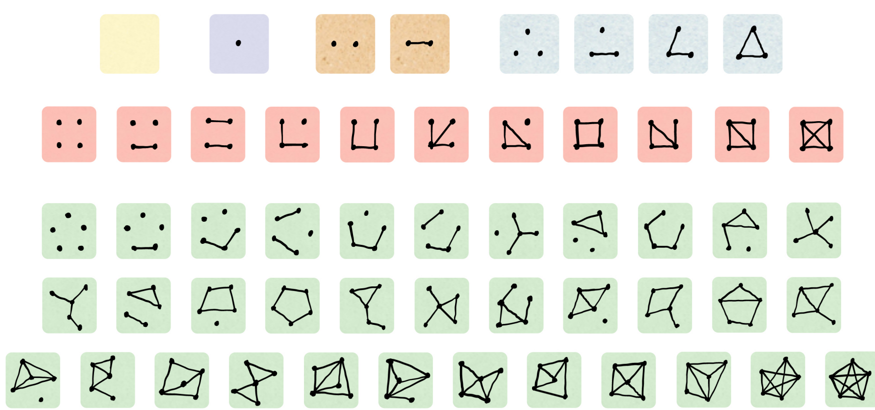
$$\sum_{i,j\in\mathbb{Z}} F_{i,j}(x)E^iN^j = \sum_{i,j\in\mathbb{Z}}\Big(\sum_{n=0}^{\infty} w_n(i,j)x^n\Big)E^iN^j = \sum_{n=0}^{\infty}\Big(\sum_{i,j\in\mathbb{Z}} w_n(i,j)E^iN^j\Big)x^n$$
$$= \sum_{n=0}^{\infty} A^n x^n = (1-Ax)^{-1}.$$

We may expand the latter expression as a sum of terms of the form $w_{i,j}(n)E^iN^jx^n$ to obtain the desired coefficients $w_{i,j}(n)$. Alternatively, using the Multinomial Theorem, we may expand

$$A^n = (E+E^{-1}+N+N^{-1})^n = \sum_{r,s,t,u}\binom{n}{r,s,t,u}E^{r-s}N^{t-u}$$

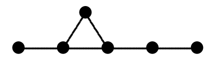
and read off the coefficient of $E^iN^j$ to obtain any desired coefficient $w_{i,j}(n)$. (In the latter sum, we take all non-negative integers $r,s,t,u$ adding up to $n$.) However, rather than explicitly expanding such a sum, it is usually preferable to allow Maple (or comparable software) to handle the expansions internally and display for us the desired coefficients, as I have shown in the example above.

5. The sequence $a_0, a_1, a_2, \ldots$ is A000088 in the OEIS. The terms for $n = 0,1,2,3,4,5$ are $a_n = 1,1,2,4,11,34$:



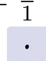
For each $n$, we found it helpful to list graphs according to the number of edges, in increasing order. This way, the list of graphs of order $n$ having $e+1$ edges makes use of the list of graphs having $e$ edges. Also it suffices to classify graphs with $e \leqslant \frac{n}{2}$ edges, then take their complements for the remaining graphs.

The number of connected graphs is $b_n = 0,1,1,2,6,21$ for $n = 0,1,2,3,4,5$. This can be counted using the list of graphs above. These form the first few terms of the sequence A000088 in the OEIS, except that OEIS uses $b_0 = 1$ and includes a discussion (with references) for the competing reasons for choosing $b_0 = 0$ or $b_0 = 1$. The question of whether the the empty graph is connected depends on which precise definition one uses. In the question I indicated that a graph is connected iff it has a vertex that is connected to all other vertices by some path; and by this definition, the empty graph is not connected, as some authors have noted. And the formula $C(x) = \ln G(x)$ (see #7) requires this interpretation. Indeed, if $C(0) = 1$ then $G(0) = e$, which would say that the total number of empty graphs is $e = 2.71728\ldots$. In effect the authors who say that the empty graph is not connected are forced to make exceptions to the rules to accommodate their choice.

6. The smallest such graph has order 6; here one can take e.g. the graph  or its complement. The fact that no graph of order $\in \{2, 3, 4, 5\}$ has such a property can be seen by glancing through the list of graphs in #1.

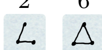
7. $c_0 = 0$

$c_1 = \frac{1}{1} = 1$

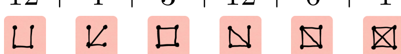


$c_2 = \frac{2}{2} = 1$

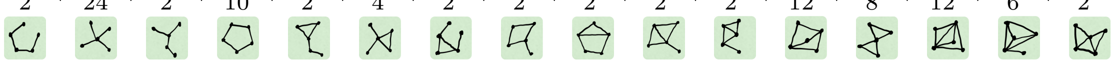
$c_3 = \frac{6}{2} + \frac{6}{6} = 3 + 1 = 4$



$$c_4 = \frac{24}{2} + \frac{24}{6} + \frac{24}{8} + \frac{24}{2} + \frac{24}{4} + \frac{24}{24}$$
$$= 12 + 4 + 3 + 12 + 6 + 1 = 38$$



$$c_5 = \frac{120}{2} + \frac{120}{24} + \frac{120}{2} + \frac{120}{10} + \frac{120}{2} + \frac{120}{4} + \frac{120}{2} + \frac{120}{2} + \frac{120}{2} + \frac{120}{2} + \frac{120}{2} + \frac{120}{12} + \frac{120}{8} + \frac{120}{12} + \frac{120}{6} + \frac{120}{2}$$



$$+ \frac{120}{4} + \frac{120}{8} + \frac{120}{4} + \frac{120}{12} + \frac{120}{120}$$

$$= 60 + 5 + 60 + 12 + 60 + 30 + 60 + 60 + 60 + 60 + 60 + 10 + 15 + 10 + 20 + 60$$
$$+ 30 + 15 + 30 + 10 + 1$$
$$= 728$$

We check these values by evaluating the first few terms of the exponential generating function $C(x) = \ln G(x)$ (see accompanying Maple worksheet). From this we see that the first few terms $c_n = 0, 1, 1, 4, 38, 728$ match our counts exactly.

8. The next three terms in the sequence are 750, 1155, 1705 as we find by completing the table of differences, and extending by three more rows under the assumption that $\Delta^4 a_n = 3$ is constant and $\Delta^5 a_n = 0$ where $a_n = \left\{ {n \atop n-2} \right\}$:

| $n$ | $a_n{=}\left\{ {n \atop n-2} \right\}$ | $\Delta a_n$ | $\Delta^2 a_n$ | $\Delta^3 a_n$ | $\Delta^4 a_n$ | $\Delta^5 a_n$ | $\Delta^6 a_n$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 3 | 0 | 0 |
| 1 | 0 | 0 | 1 | 4 | 3 | 0 | 0 |
| 2 | 0 | 1 | 5 | 7 | 3 | 0 | 0 |
| 3 | 1 | 6 | 12 | 10 | 3 | 0 | 0 |
| 4 | 7 | 18 | 22 | 13 | 3 | 0 | 0 |
| 5 | 25 | 40 | 35 | 16 | 3 | 0 | 0 |
| 6 | 65 | 75 | 51 | 19 | 3 | 0 | 0 |
| 7 | 140 | 126 | 70 | 22 | 3 | 0 | |
| 8 | 266 | 196 | 92 | 25 | 3 | | |
| 9 | 462 | 288 | 117 | 28 | | | |
| 10 | 750 | 405 | 145 | | | | |
| 11 | 1155 | 550 | | | | | |
| 12 | 1705 | | | | | | |

9. Using Pascal's relation $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$, we have $\Delta\binom{n}{k} = \binom{n+1}{k} - \binom{n}{k} = \binom{n}{k-1}$.

10. Denoting $a_n = \left\{ {n \atop n-2} \right\}$, we have $a_n = 1\binom{n}{3} + 3\binom{n}{4} = \frac{1}{24}n(n-1)(n-2)(3n-5)$. We verify that this polynomial fits the known values of $\left\{ {n \atop n-2} \right\}$ exactly for $n = 0, 1, 2, \ldots, 12$.

11. This is essentially Problem #4(c) on HW4, but we have spelled out the steps (see Solutions to HW4). You should verify that the number of surjections $[n] \to [n{-}2]$ is $\left\{ {n \atop n-2} \right\}(n-2)! = \frac{1}{24}n(n-1)(n-2)(3n-5)(n-2)! = \frac{1}{24}(n-2)(3n-5)n!$, in agreement with our answer there.