University of Wyoming Math 3700—Spring 2023 Combinatorics Department of Mathematics

Practice Problems

May 2023

1. (a) How many ways can I give eleven identical coins to eight students?

   (b) How many ways can I give eleven identical coins to eight students, if each student is expected to receive at least one of the coins?

   (c) How many ways can I give eleven different coins to eights students?

   (d) How many ways can I give eleven different coins to eights students, if each student is expected to receive at least one of the coins?

2. Let $S(x) = \sum_{n=0}^{\infty} n^2 x_n$, the generating function of the sequence of square numbers $0, 1, 4, 9, 16, 25, \ldots$. Evaluate $S(x)$ as a rational function in simple closed form.

3. Let n be a non-negative integer. Find closed formulas for

   (a) $\sum_{k=0}^{n} \binom{n}{k} k$ (b) $\sum_{k=0}^{n} \binom{n}{k} k^2$ (c) $\sum_{k=0}^{n} \binom{n}{k}^2$

4. A city has streets running north-south and avenues running east-west, forming a grid in which all blocks have the same size.

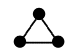
   (a) If I am at a particular intersection and I need to walk 12 blocks east and 5 blocks north using a path of shortest distance possible, I will need to walk 17 blocks. How many possible routes are there which allow me to reach my destination in just 17 blocks?

   *(b) Mark each intersection by its coordinates (i, j) with respect to the main avenue as x-axis and the main street as y-axis; for example the intersection $(3, -8)$ is three blocks north and eight blocks south of intersection $(0, 0)$. Let $w_n(i, j)$ be the number of walks of length n from intersection $(0, 0)$ to intersection (i, j). (A walk is any route formed by following city blocks, not necessarily distinct blocks. When counting walks, the order in which the blocks are followed matters.) For each (i, j), compute the generating function $F_{i,j}(x) = \sum_{n=0}^{\infty} w_n(i, j) x^n$. Use this to determine $w_n(i, j)$ and give another method for answering (a).

5. Consider the sequence a_n defined as the number of isomorphism classes of ordinary graphs (undirected graphs with no loops or multiple edges) of order n. Also define b_n to be the number of isomorphism classes of connected graphs of order n. Tabulate the values of a_n and b_n for $n = 0, 1, 2, 3, 4, 5$. (For technical reasons, we should have

$a_0 = 1$ and $b_0 = 0$. To make sense of this, consider the following: A graph is connected iff it has a vertex which is joined to every vertex by some path.)

6. An *automorphism* of a graph is an isomorphism from the graph to itself. What is the smallest nontrivial graph (i.e. having more than one vertex) having no automorphism other than the trivial one (the identity automorphism mapping each vertex to itself)? (Again, we only consider ordinary graphs here.)

In #5, we were of course counting unlabelled graphs. The number of *labelled* graphs of order n is simply $2^{\binom{n}{2}}$, assuming we use the set $[n] = \{1, 2, \ldots, n\}$ as the set of labels. Given a labelled graph Γ of order n, the number of labelled graphs (having label set $[n]$) to which it is isomorphic, is $\frac{n!}{|\operatorname{Aut}\Gamma)|}$. In other words, given an unlabelled graph Γ of order n, there are $\frac{n!}{|\operatorname{Aut}\Gamma)|}$ labelled graphs isomorphic to Γ. For example, the triangle  has 6 automorphisms, and it corresponds to just $\frac{3!}{6} = 1$ labelled graph; every labelling using the label set $[3] = \{1, 2, 3\}$ gives the same labelled graph. The unlabelled graph has 2 automorphisms, and it corresponds to $\frac{3!}{2} = 3$ different labelled graphs (there are three choices for which label to place on the middle vertex). So while there are 2 unlabelled graphs of order 3 (up to isomorphism), there are altogether $1 + 3 = 4$ labelled connected graphs of order 4.

7. Enumerate the number of *labelled* connected graphs c_n for $n = 1, 2, 3, 4, 5$. (For technical reasons, $c_0 = 0$; also $c_3 = 4$ as explained above. This leaves the cases $n = 1, 2, 4, 5$ for you to figure out, using your lists of small graphs in #5.)

The exponential generating function for the number of *labelled* graphs of order n is

$$G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{x^n}{n!} = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{8}{3}x^4 + \frac{128}{15}x^5 + \frac{2048}{45}x^6 + \frac{131072}{315}x^7 + \cdots.$$

Standard counting techniques show that the exponential generating function for c_n, the number of labelled connected graphs (see #7) is

$$C(x) = \ln G(x).$$

It would be very useful to have explicit formulas for the number of unlabelled graphs a_n, also for the number of unlabelled connected graphs b_n; however it is very hard to relate this to c_n since unlabelled graphs vary greatly in terms of the size of their automorphism groups.

Next, we continue the theme of *finite differences* begun in HW4. Given a sequence $(a_0, a_1, a_2, \ldots)$, its sequence of *forward differences* is

$$\Delta(a_0, a_1, a_2, \ldots) = (a_1 - a_0,\ a_2 - a_1,\ a_3 - a_2,\ \ldots).$$

The *second order differences* form the sequence

$$\Delta^2(a_0, a_1, a_2, \ldots) = \Delta(\Delta(a_0, a_1, a_2, \ldots)) = (a_2 - 2a_1 + a_0,\ a_3 - 2a_2 + a_1,\ a_4 - 2a_3 + a_2,\ \ldots).$$

Higher order differences are defined by iterating the map Δ; thus the $(n+1)$-st order differences form the sequence $\Delta^{k+1}(a_0, a_1, a_2, \ldots) = \Delta(\Delta^k(a_0, a_1, a_2, \ldots))$. We also write $\Delta a_n = a_{n+1} - a_n$ for term n in the sequence of first-order differences; similarly $\Delta^k a_n$ denotes term n in the sequence of k-th order differences. For example, $\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n$ and $\Delta^3 a_5 = a_8 - 3a_7 + 3a_6 - a_5$.

Finite differences are a very handy tool for studying sequences. For example, what is the next term in the sequence $3, 4, 9, 20, 39, 68, 109, \ldots$? From the data, we generate a table of values for $n, a_n, \Delta a_n, \Delta^2 a_n, \Delta^3 a_n, \ldots$:

n	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$	$\Delta^4 a_n$	$\Delta^5 a_n$	$\Delta^6 a_n$
0	3	1	4	2	0	0	0
1	4	5	6	2	0	0	
2	9	11	8	2	0		
3	20	19	10	2			
4	39	29	12				
5	68	41					
6	109						

In general, there is no reason to assume that this pattern continues throughout the rest of the table. But if it does, then this allows us to extrapolate the sequence to find additional terms, as many terms as we like; for example in this case, we obtain

n	a_n	Δa_n	$\Delta^2 a_n$	$\Delta^3 a_n$	$\Delta^4 a_n$	$\Delta^5 a_n$	$\Delta^6 a_n$
0	3	1	4	2	0	0	0
1	4	5	6	2	0	0	0
2	9	11	8	2	0	0	0
3	20	19	10	2	0	0	0
4	39	29	12	2	0	0	0
5	68	41	14	2	0	0	0
6	109	55	16	2	0	0	0
7	164	71	18	2	0	0	0
8	235	89	20	2	0	0	0
9	324	109	22	2	0	0	0

We have extrapolated the sequence to find the next three terms 164, 235, 324, ... and we can in fact extrapolate the sequence as far as we want. Of course the validity of this extrapolation depends on the validity of our assumption that $\Delta^3 a_n = 2$, a constant value for all n; and $\Delta^4 a_n = 0$ for all n. In general, if $\Delta^k a_n$ is constant and $\Delta^{k+1} a_n = 0$, then a_n is a polynomial in n of degree k, and this method works. We will explain these conditions below; but you should think of them as the discrete version of the conditions that $\frac{d^k}{dx^k} f(x) = \text{constant}$ and $\frac{d^{k+1}}{dx^{k+1}} f(x) = 0$, these being the conditions for $f(x)$ to be a polynomial of degree k. For most data, unless it represents values of a polynomial, it is not possible to extrapolate reliably from a given finite set of values of the sequence.

8. In class we gave the first few rows of Stirling's Triangle. From there, we can read off the first few terms of the sequence $\left\{ {n \atop n-2} \right\} = 0, 0, 0, 1, 7, 25, 65, 140, 266, 462, \ldots$ for $n =$

$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$. Find the next three terms in the sequence *by extrapolating, using a table of finite differences as we have done in the example above.*

In the example above, we showed how to extrapolate data given by a polynomial function evaluated at 0, 1, 2, 3, $\ldots$, m. We did not explain why this works; nor did we explain how to produce the polynomial defining the data. This is explained in a sequence of steps, beginning with the following exercise. Recall that $\binom{n}{k} = \frac{P(n,k)}{k!}$.

9. Fix a non-negative integer k, and consider the sequence $\binom{n}{k}$ with terms $\binom{0}{k}$, $\binom{1}{k}$, $\binom{2}{k}$, ... Show that $\Delta\binom{n}{k} = \binom{n}{k-1}$.

Now if a_n is given by a polynomial function of degree k in n, it is just as well given by a linear combination of the polynomials $\binom{n}{i} = \frac{P(n,i)}{i!}$ for $i = 0, 1, 2, \ldots, k$. (Recall our earlier discussion that the vector space of polynomials in x has several bases. In addition to the standard basis formed by the monomials x^i, we also have the basis consisting of polynomials $P(x, i)$; and these can be divided by $i!$ to obtain a new basis consisting of the binomial coefficients $\binom{x}{i}$.)

Returning to our example above, we had a sequence defined by a polynomial of degree 3 which may be written as $a_n = \sum_{i=0}^{3} c_i \binom{n}{i}$. By repeatedly taking finite differences, we have

$$\begin{aligned} a_n &= c_0 + c_1 n + c_2\binom{n}{2} + c_3\binom{n}{3} \\ \Delta a_n &= c_1 + c_2 n + c_3\binom{n}{2} \\ \Delta^2 a_n &= c_2 + c_3 n \\ \Delta^3 a_n &= c_3 \end{aligned}$$

Now evaluating at $n = 0$ and reading off values of $c_k = \Delta^k a_0$ from our table of finite differences, we obtain $c_0{=}a_0{=}3$, $c_1{=}\Delta a_0{=}1$, $c_2{=}\Delta^2 a_0{=}4$, $c_3{=}\Delta^2 a_0{=}2$. Thus

$$\begin{aligned} a_n &= 3 + n + 4\binom{n}{2} + 3\binom{n}{3} = 3 + n + 4\cdot\tfrac{n(n-1)}{2} + 2\cdot\tfrac{n(n-1)(n-2)}{6} \\ &= \tfrac{1}{3}n^3 + n^2 - \tfrac{1}{3}n + 3. \end{aligned}$$

This method of fitting a polynomial formula for a_n from the data is known as *Newton Interpolation.*

10. Using your table of differences from #8, use the method of Newton Interpolation (explained above) to express $\left\{ {n \atop n-2} \right\}$ as a polynomial function in n.

11. Use the following combinatorial reasoning to count the number of ways to partition n distinct objects into $n{-}2$ nonempty parts.
    (a) How many partitions have one part of size 3 and $n{-}3$ parts of size 1?
    (b) How many partitions have two parts of size 2 and $n{-}4$ parts of size 1?
    (c) Add your answers in (a) and (b) and simplify, to express $\left\{ {n \atop n-2} \right\}$ as a polynomial in n. Be sure to compare with your answer in #10.