University of Wyoming — Math 3700—Spring 2023 — Combinatorics — Department of Mathematics

Solutions to HW4

1. (a) There are $P(9,5) = 9{\cdot}8{\cdot}7{\cdot}6{\cdot}5 =$ 15,120 ways. (Listing the recipient of each book gives a sequence with no repetitions.)

   (b) There are $9^5 =$ 59,049 ways (9 choices for each book, giving a sequence with possible repetitions; each outcome is an arbitrary function from the set of 5 books to the set of 9 students.)

2. (a) There are $\binom{9}{5} = 126$ ways (the number of 5-subsets of a 9-set).

   (b) There are $\binom{9+5-1}{5} = \binom{13}{5} = 1287$ ways (the number of 5-element multisets chosen from a 9-set).

3. (a) $\int_0^x F(t)\,dt = \int_0^x \frac{dt}{\sqrt{1-4t}} = -\frac{1}{2}\sqrt{1-4t}\,\Big|_0^x = \frac{1}{2}\left[1-\sqrt{1-4x}\right]$.

   (b) $\int_0^x F(t)\,dt = \sum_{n=0}^{\infty}\binom{2n}{n}\int_0^x t^n\,dt = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{n+1}}{n+1} = x\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$.

   (c) Noting that (a) and (b) must agree, we simply divide both sides by x to obtain $\frac{1}{2x}\left[1-\sqrt{1-4x}\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n$.

4. (a) $n!$

   (b) There are $n{-}1$ ways to choose which value j in the range $[n{-}1]$ occurs twice as a function value, then $\binom{n}{2}$ ways to choose which elements of the domain map to j, then $(n-2)!$ ways to chose how to map the remaining $n{-}2$ elements of the domain onto the remaining $n{-}2$ elements of the range. Altogether this gives $(n{-}1)\binom{n}{2}(n{-}2)! = \frac{(n-1)n!}{2}$.

   This answer can also be written as $\left\{ {n \atop n-1} \right\}(n-1)!$ in the notation of Stirling numbers, which is correct but less explicit: one could evaluate $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$, leading to the same answer.

   (c) There are two possibilities: either (i) some value j in the range $[n{-}2]$ occurs three times, or (ii) two values $j \neq j'$ in the range $[n{-}2]$ each occur twice.

   In case (i), there are $n{-}2$ ways to choose $j \in [n{-}2]$, then $\binom{n}{3}$ ways to choose which elements of the domain map to j, then $(n{-}3)!$ ways to map the remaining $n{-}3$ elements of the domain onto the remaining $n{-}3$ elements of the range. So there are $(n{-}2)\binom{n}{3}(n{-}3)! = \binom{n}{3}(n{-}2)! = \frac{(n-2)n!}{6}$ ways in case (i).

   In case (ii), there are $\binom{n-2}{2}$ ways to choose distinct values $j \neq j'$ in the range $[n{-}2]$, then $\binom{n}{2}$ ways to choose which elements of $[n]$ to map to j, then $\binom{n-2}{2}$ ways to choose which of the remaining elements to map to j'; and finally, $(n{-}4)!$ ways to map the remaining $n{-}4$ elements of the domain onto the remaining

$n-4$ elements of the range. This makes $\binom{n-2}{2}^2\binom{n}{2}(n-4)! = \frac{(n-2)(n-3)n!}{8}$ ways in case (ii).

Combining cases (i) and (ii), the total number of surjections $[n] \to [n-2]$ is

$$\frac{(n-2)n!}{6} + \frac{(n-2)(n-3)n!}{8} = \frac{(n-2)(3n-5)n!}{24}.$$

Once again, the answer $\left\{ {n \atop n-2} \right\}(n-2)!$ is correct, although less explicit.

(d) $2^n - 2$, if $n \geqslant 2$; otherwise 0. There are 2^n functions $[n] \to [2]$, but we must exclude the two constant functions.

5. (a) $D(x) = \sum_{n=0}^{\infty}(a_{n+1} - a_n)x^n = \sum_{n=0}^{\infty} a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n = \frac{1}{x}\sum_{n=1}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n$

$$= \frac{1}{x}\big[A(x) - A(0)\big] - A(x) = \frac{1-x}{x}A(x) - \frac{1}{x}A(0).$$

(b) $A(x) = \sum_{n=0}^{\infty}\binom{n+7}{7}x^n = (1-x)^{-8}$. (In class we obtained the coefficients a_n from this series.)

(c) $D(x) = \frac{1-x}{x}(1-x)^{-8} - \frac{1}{x} = \frac{1}{x}\big[(1-x)^{-7} - 1\big]$.

Check: $A(x) = 1 + 8x + 36x^2 + 120x^3 + 330x^4 + 792x^5 + \cdots$

$D(x) = 7 + 28x + 84x^2 + 210x^3 + 462x^4 + \cdots$

6. (a) There are $5^{22} = 2{,}384{,}185{,}791{,}015{,}625$ possible outcomes (the number of arbitrary functions from the set of 22 students to the set of 5 possible letter grades).

(b) There are $\binom{22}{5,7,7,2,1} = \frac{22!}{5!7!7!2!1!} = 184{,}371{,}707{,}520$ such outcomes (the number of 22-letter words containing five A's, seven B's, seven C's, two D's and one F).