UNIVERSITY OF WYOMING

Math 3700—Spring 2023

Combinatorics

Department of Mathematics

Solutions to HW3

1. (a) The girth of Γ is 4. There are no 3-cycles in Γ; but there exists a pair of vertices x, y at distance 2 with two common neighbors z, w, so there is a 4-cycle $x \sim z \sim y \sim w \sim x$.

   (b) Fix a vertex, which we label 0. This vertex has k neighbors which we may label $1, 2, \ldots, k$. There are no edges between any two vertices of $V_1 = \{1, 2, \ldots, k\}$; so each $i \in V_1$ has $k{-}1$ neighbors in V_2, the set of vertices at distance 2 from vertex 0. Since Γ has diameter 2, its vertex set is $\{0\} \sqcup V_1 \sqcup V_2$ and $n = 1{+}k{+}|V_2|$. Again, since each vertex in V_1 has $k{-}1$ neighbors in V_2, there are exactly $k(k{-}1)$ edges between V_1 and V_2. Also by hypothesis, each vertex in V_2 has 2 neighbors in V_1, so the number of edges between V_1 and V_2 is $2|V_2|$. Since $2|V_2| = k(k{-}1)$, we have $|V_2| = k(k{-}1)/2$ and $n = 1 + k + (k(k{-}1)/2 = \frac{1}{2}\big(k^2{+}k{+}2\big)$.

   (c) The entries of A^2 count the walks of length 2 between pairs of vertices of Γ. All entries on the main diagonal of A^2 are k, since Γ is k-regular; this gives kI_n as one term in A^2. If $x \sim y$, then there are no walks of length 2 from x to y; so the (x, y)-entry of A^2 is zero. If $x \not\sim z$, then there are two walks of length 2 from x to z, so the (x, z)-term of A^2 is 2, This gives a final term $2(J{-}I{-}A)$ in A^2; so

   $$A^2 = kI + 2(J{-}I{-}A).$$

   (d) The all-ones vector $\mathbf{j}$ of size $n \times 1$ satisfies $A\mathbf{j} = k\mathbf{j}$. Now consider any eigenvector $\mathbf{u}$ orthogonal to $\mathbf{j}$, having eigenvalue λ; then $\lambda^2\mathbf{u} = A^2\mathbf{u} = k\mathbf{u} + 2(J{-}I{-}A)\mathbf{u} = k\mathbf{u} - 2\mathbf{u} - 2\lambda\mathbf{u}$ so

   $$\lambda^2 + 2\lambda - (k{-}2) = 0.$$

   This gives $\lambda = \frac{1}{2}\big({-2} \pm \sqrt{4k{-}4}\big) = -1 \pm \sqrt{k{-}1} < k$. This tells us a couple things. First of all, the eigenvalue k (from the eigenvector $\mathbf{j}$) has multiplicity 1; and secondly, the only remaining eigenvalues are $\lambda = -1{+}\sqrt{k{-}1}$ and $\mu = -1{-}\sqrt{k{-}1}$, with certain multiplicities m and $n{-}m{-}1$ respectively. Since $\mathrm{tr}(A) = 0$, we have the sum of its eigenvalues (counting multiplicity) is also zero, i.e.

   $$k + m\lambda + (n{-}m{-}1)\mu = 0.$$

   As previously noted, the eigenvalue k has multiplicity 1. Since $n = \frac{1}{2}\big(k^2{+}k{+}2\big)$, we find that the eigenvalue $\lambda = -1 + \sqrt{k{-}1}$ has multiplicity

   $$m = \tfrac{1}{4}k\big(k + 1 + \sqrt{k-1}\big).$$

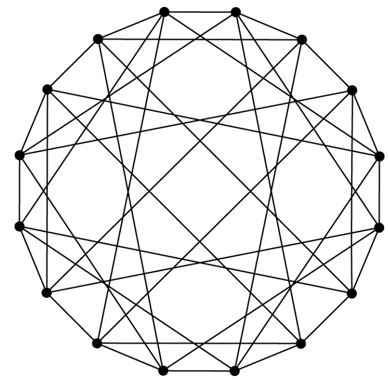
This also means that the remaining eigenvalue $\mu = -1 - \sqrt{k-1}$ has multiplicity

$$n-m-1 = \tfrac{1}{4}k\big(k+1-\sqrt{k-1}\,\big).$$

Note that these formulas require $k-1$ to be a perfect square. Other than the trivial example with $k=1$, $n=4$ (a 4-cycle), we have $k=5$, $n=16$ which gives the example in (e).

(e) For $k=5$, we obtain $n=16$, and Γ has spectrum
  $k=5$, with multiplicity 1;
  $\lambda = 1$, with multiplicity 10; and
  $\mu = -3$, with multiplicity 5.

This gives the Clebsch graph, shown on the right.



2. (a) You should be able to check at least the first half of these values by hand; after that probably rely on the recurrence relation or the computer expansion of (c).

n	0	1	2	3	4	5	6	7	8	9	10
w_n	1	1	1	3	7	13	25	51	103	205	409

(b) $w_n = 1$ for $n \in \{0,1,2\}$ and $w_n = 2w_{n-1} - w_{n-2} + 2w_{n-3}$ for $n \in \{3,4,5,6,\dots\}$. This follows directly from the denominator in (c). Alternatively, since the graph has only 3 vertices, we know the recurrence ls linear of depth 3, and the first six terms of the sequence in (a) give us a linear system which can be solved uniquely to obtain our linear recurrence. (And a few more terms in the to check that our solution is correct.)

(c) Using Maple, we find that the $(1,1)$-entry of $(I-xA)^{-1}$ is

$$W(x) = \frac{1-x}{1-2x+x^2-2x^3}.$$

(d) Again using Maple,

$$W(x) = 1 + x + x^2 + 3x^3 + 7x^4 + 13x^5 + 25x^6 + 51x^7 + 103x^8 + 205x^9 + 409x^{10} + 819x^{11} + 1639x^{12} + 3277x^{13} + 6553x^{14} + 13107x^{15} + \cdots.$$

(e) $W(x) = \frac{1-x}{1-2x+x^2-2x^3} = \frac{A}{1-2x} + \frac{Bx+C}{1+x^2}$ for certain real constants A, B, C which we must determine. Clearing denominators, we obtain

$$1 - x = A(1+x^2) + (Bx+C)(1-2x).$$

Evaluating at $\frac{1}{2}$ and at $\pm i$, we obtain linear equations which we solve to obtain $A{=}\frac{2}{5}$, $B{=}\frac{1}{5}$, $C{=}\frac{3}{5}$; so

$$\begin{aligned} W(x) &= \frac{2}{5(1-2x)} + \frac{3+x}{5(1+x^2)} \\ &= \tfrac{2}{5}\big(1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots\big) + \tfrac{1}{5}\big(3+x\big)(1 - x^2 + x^4 - x^6 + \cdots). \end{aligned}$$

Reading off the coefficient of x^n, $w_n = \begin{cases} \frac{1}{5}\big(2^{n+1}+3(-1)^{n/2}\big), & \text{if } n \text{ is even;} \\ \frac{1}{5}\big(2^{n+1}+(-1)^{(n-1)/2}\big), & \text{if } n \text{ is odd.} \end{cases}$

(f) The dominant term in our expression for w_n gives $w_n \sim \frac{2}{5}\cdot 2^n$. Check by comparison:

n	0	1	2	3	4	5	6	7	8	9	10
w_n	1	1	1	3	7	13	25	51	103	205	409
$\frac{2}{5}\cdot 2^n$	0.4	0.8	1.6	3.2	6.4	12.8	25.6	51.2	102.4	204.8	409.6