University of Wyoming — Math 3700—Spring 2023 — Combinatorics — Department of Mathematics

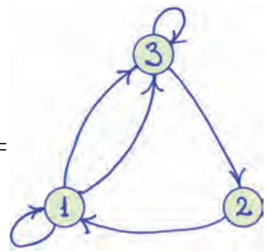
HW3

(Due 5:00 pm, Wednesday, April 19, 2023 *on WyoCourses*)

Instructions: See the syllabus for general instructions for completing homework. Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using complete sentences where appropriate, and always using correct notation. Note the correction in #1, indicated in red; without this you will not be able to answer the problem. As a result of this correction, I am extending the due date to April 19, as indicated above (the original due date was a week earlier).

1. Refer to the handout on Moore Graphs. Let Γ be a finite undirected k-regular graph of order n with no loops or multiple edges. Assume Γ has no triangles. Also assume Γ has diameter 2, and any two vertices at distance 2 have exactly two common neighbors. (Thus if $x \neq y$ and $x \not\sim y$, then there exist exactly two vertices z satisfying $x \sim z \sim y$.)
   (a) *(5 points)* What is the girth of Γ?
   (b) *(10 points)* Express n as a polynomial of degree 2 in k, using an argument similar the argument we used for Moore graphs.
   (c) *(10 points)* Let A be the adjacency matrix of Γ, and let J and I be the all-ones matrix and identity matrix, respectively, all of size $n \times n$. Express A^2 as a linear combination of A, J and I as we did for Moore graphs.
   (d) *(10 points)* Using (c), determine the eigenvalues of A, and their multiplicities (i.e. the number of times each eigenvalue occurs), just as we did for Moore graphs. These quantities will depend on k.
   (e) *(5 points)* Finally, assuming $k = 5$, determine the eigenvalues of A and their multiplicities.

2. Let Γ be the graph shown, along with its adjacency matrix A. Note that Γ is a directed graph having loops and multiple edges.

$\Gamma =$ 

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer the following, using a combination of hand computations and computer algebra packages (with appropriate explanations).

(a) *(10 points)* Let $w_n = w_n(1,1)$ be the number of walks of length n from vertex 1 to itself. Tabulate the exact values of w_n for $n = 0, 1, 2, \ldots, 10$.

(b) *(10 points)* Find a linear recurrence of depth 3 satisfied by the sequence (w_n), and use it to express the recursive definition of this sequence.

(c) *(10 points)* Using the method discussed in class (and in the accompanying handout), express the generating function $W(x) = \sum_{n=0}^{\infty} w_n x^n$ as a rational function of x, in simplest form.

(d) *(5 points)* Find the first 15 terms in the series expansion of $W(x)$. Check by comparing with your answer in (a).

(e) *(10 points)* Find the partial fraction decomposition of $W(x)$ and use this to give an exact closed formula for w_n.

(f) *(5 points)* Using (e), find an asymptotic formula $w_n = A\alpha^n$ where A and α are explicit constants, thus showing that w_n grows exponentially. To check, see how well your asymptotic estimate does in approximating w_{10}.