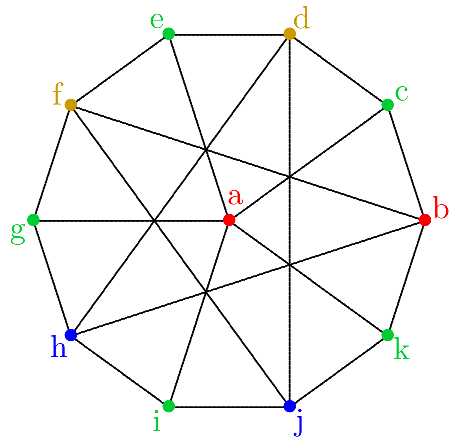
Solutions to HW2

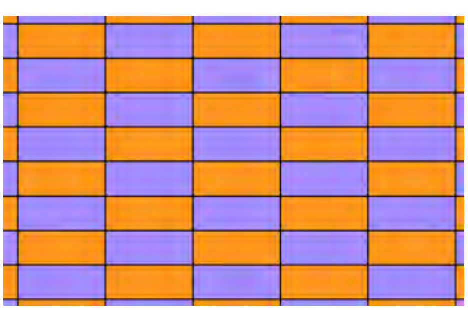
1. Denote the given graph by Γ, and label its vertices as shown. This graph is known as the *Grötzsch graph*.



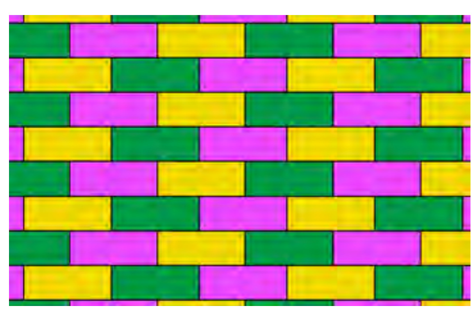
   (a) The maximum number of classes whose exams can be given simultaneously is the independence number $\alpha(\Gamma) = 5$. The green vertices {c,e,g,i,k} form a 5-coclique, and we claim that this is maximal. No maximal coclique can contain the central vertex 'a', as such a coclique contains none of the green vertices c,e,g,i,k and at most two of the vertices in the remaining 5-cycle b,f,j,d,h. So any maximal clique must be contained in the circuit of length 10 formed by the outer vertices b,c,d,e,f,g,h,i,j,k. No two consecutive vertices in this circuit are allowed, so every coclique has at most 5 vertices.

   (b) The minimum number of two-hour time slots required to schedule all the exams without any conflicts is the chromatic number $\chi(\Gamma) = 4$. We have shown above how to proper 4-color the vertices, so $\chi(\Gamma) \leqslant 4$. Now we must show that it is not possible to properly 3-color the vertices. Suppose there is a proper 3-coloring of the vertices of Γ, say with colors red, blue, brown. The 5-cycle b,f,j,d,h requires all five colors; and by symmetry, we may suppose that b is red, d and f are brown, and j and h are blue as shown above. But this forces c to be blue, g to be red, and k to be brown; then no color is available for 'a', a contradiction.

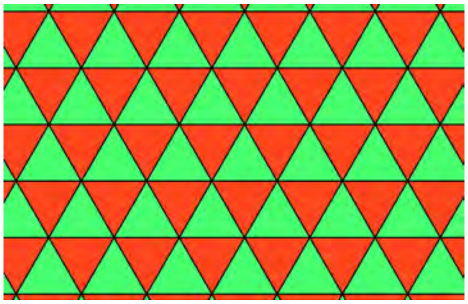
2. (a) 2 colors are required



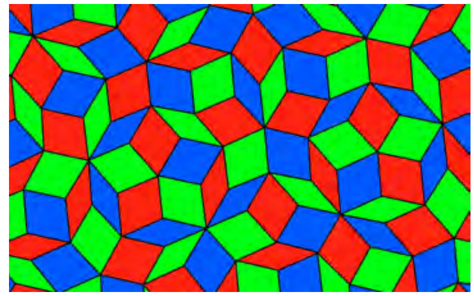
   (b) 3 colors are required



(c) 2 colors are required



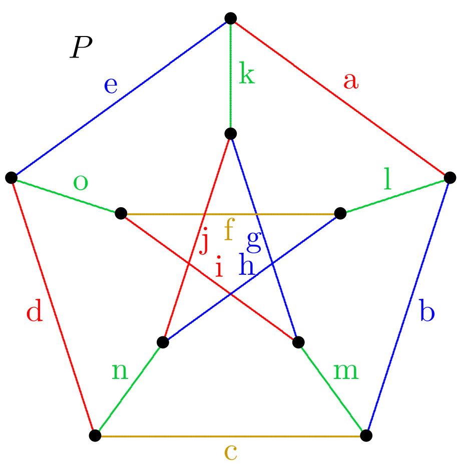
(d) 3 colors are required

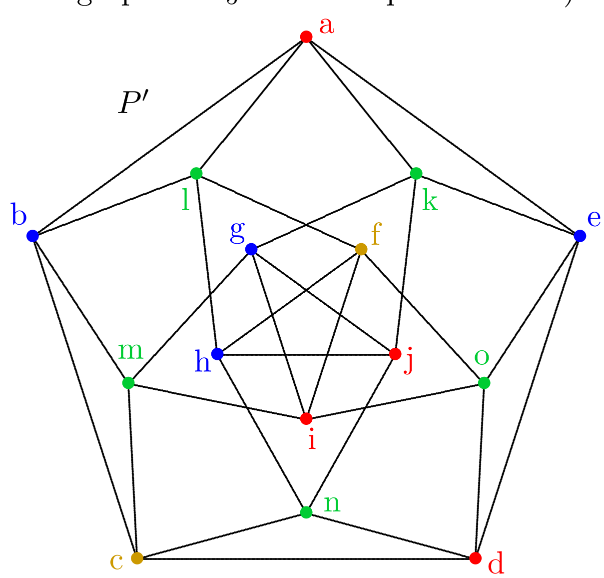


The map in (d) is a portion of the *rhombic Penrose tiling* of the plane. There are *many* ways to properly 3-color this portion; and if you follow the hint, you should succeed in finding a 3-coloring like the one shown. This pattern extends to a tiling of the entire plane using the two rhombic tile shapes shown; and this tiling is aperiodic (it does not repeat in the way that the tilings (a), (b), (c) do). Conway conjectured that this entire tiling is properly 3-colorable, and this was later proved by Sibley and Wagon (2000).

3. The graph graph P' is known as the *line graph of* P. We can, in a similar way, construct the line graph of any graph (for example, the line graph of K_5 is the complement of P).

(a)





(b) The diameter of P' is 3. For example, $d(a, n) = 3$. The vertex set can be partitioned as $V = V_1 \sqcup V_2 \sqcup V_3$ where $V_1 = \{$a,b,c,d,e$\}$ (the outer vertices), $V_2 = \{$f,g,h,i,j$\}$ (the inner vertices), and $V_3 = \{$k,l,m,n,o$\}$ (the middle vertices). Any two vertices in the same part V_i are at distance at most 2. And every vertex in one part has neighbors in the other two parts. This means that any two vertices are joined by a path of length at most 3.

(c) The clique number $\omega(P') = 3$, and $\{$a,e,k$\}$ is an example of a 3-clique in P'. Every 3-clique in P' corresponds to a set of 3 edges in P a common vertex in P.

This is the only way for three edges in P to all touch each other, since there are no triangles in P. And since each vertex in P has only 3 edges, there are no larger sets of 4 or more edges in P that all touch each other (hence no 4-cliques in P').

(d) The coclique number (i.e. independence number) is $\alpha(P') = 5$. In the illustration above, the green inner vertices $V_3 = \{$k,l,m,n,o$\}$ form a 5-coclique (independent set of size 5). There is no larger coclique in P' than this, since a coclique in P' corresponds to a set of edges in P with no two touching each other. Since there are only 10 vertices in P, every such set of edges has size at most $\frac{10}{2} = 5$.

(e) We show that the chromatic number $\chi(P') = 4$. In (a) we have provided a proper vertex coloring of P' with 4 colors, so $\chi(P') \leqslant 4$. This was obtained by coloring all vertices in the inner coclique V_2 green. This leaves only two 5-cycles, each of which can be properly colored using 3 other colors.

One cannot do better. If there were a proper coloring of the vertices of P' using only three colors, say red, green and blue, then we would need five vertices of each color (the maximum size of a coclique, by (d)). The green vertices in P' would corresponding to five edges in P with no two touching. The remaining edges in P form a 2-regular subgraph passing through every vertex of P. Such a subgraph is a collection of cycles. Since P has no cycles of length less than 5, and P has no Hamilton circuit (10-cycle passing through all vertices), this means we are necessarily left with two 5-cycles as considered above; and these require three additional colors.