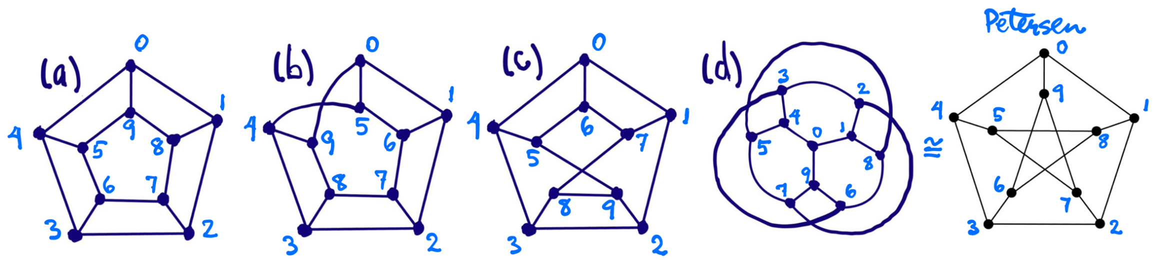
Solutions to HW1

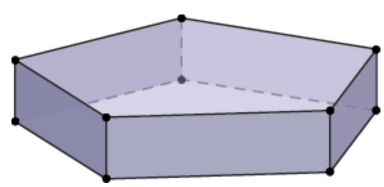
1. No two of the graphs are isomorphic. There are many ways to see this; for example the number of 4-cycles in (a), (b), (c), (d) is 5, 5, 3, 0 respectively. Also (a) has two 5-cycles but (b) has no 5-cycles. Graph (a) is clearly planar; but it can be shown that each of the graphs (b), (c), (d) is nonplanar. Graph (b) is bipartite; none of the others is bipartite.

   The graphs (a), (b), (c), (d) have exactly 20, 20, 4, 120 automorphisms respectively. I will describe each of the automorphism groups in turn, first in words and then using notation for permutation groups, where we use vertex labels as follows:



   Students who have not studied permutation groups can safely ignore the second, more technical description.

   Graph (a) can be viewed as the edges of a solid prism with two pentagonal faces (top and bottom) and five rectangular faces (on the sides):



   This prism has 10 rotational symmetries; but its symmetry group is twice as large due to the existence of reflective symmetries. The automorphism group is

$$\operatorname{Aut}\Gamma_{(a)} = \big\langle (01234)(59876),\ (14)(23)(58)(67),\ (09)(18)(27)(36)(45) \big\rangle \cong D_5 \times C_2$$

   of order 20 (a direct product of a dihedral group of order 10 with a cyclic group of order 2).

   Graph (b) has five 4-cycles strung together as a Möbius strip. Every automorphism of the graph must preserve this arrangement while also preserving the 10-cycle $(0,1,2,3,4,5,6,7,8,9,0)$, the boundary of the Möbius strip. This boundary has 20 symmetries forming a dihedral group of order 20:

$$\operatorname{Aut}\Gamma_{(b)} = \big\langle (0123459876),\ (14)(23)(69)(78) \big\rangle \cong D_{10}\,.$$

Graph (c) has exactly three 4-cycles, two of which are joined along the edge $\{0,6\}$, and the other having vertices $2,3,8,9$. Every automorphism of the graph must preserve these distinct features, so

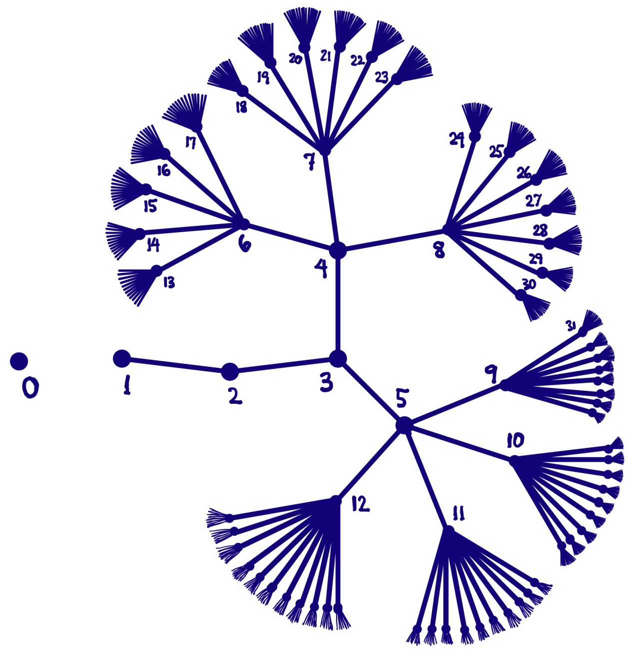
$$\mathrm{Aut}\,\Gamma_{(c)} = \big\langle (14)(23)(57)(89),\ (06)(17)(28)(39)(45) \big\rangle \cong C_2 \times C_2\,,$$

a Klein 4-group.

Graph (d) is a Petersen graph, as shown by the isomorphism appearing above. Thus

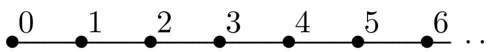
$$\mathrm{Aut}\,\Gamma_{(d)} = \big\langle (01234)(59876),\ (149)(256)(378),\ (19)(27)(35)(68) \big\rangle \cong S_5$$

of order 120, a symmetric group of degree 5.



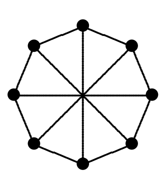
2. (a) No; in the infinite graph shown on the right, no two vertices have the same degree. In fact, every integer $i \geqslant 0$ is the degree of exactly one vertex. We label each vertex by its degree. Of course we have not drawn the entire (infinite) graph, only enough vertices so that you can understand the pattern by which the graph is constructed.

   (b) Yes. The graph shown in (a) is an example; but a simpler example is the infinite path

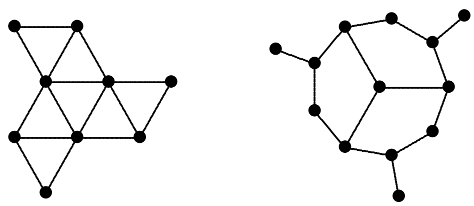
   in which each vertex is labelled by its distance from the vertex 0 (the unique vertex of degree 1). Since no two vertices have the same distance from the unique vertex of degree 1, clearly this graph also has no nontrivial automorphisms.

3. (a) There exists a 3-regular graph of order n iff $n \in \{4,6,8,10,12,\ldots\}$, i.e. $n > 3$ and n is even. This condition is necessary since the sum of the vertex degrees is $nk = 2e$, an even number, where e is the number of edges. The condition is also sufficient since for every even number $n \geqslant 4$, we can form a 3-regular graph of order n by starting with an n-cycle (n-gon) and adding all $\frac{n}{2}$ of its 'main diagonals' (joining each vertex with the opposite vertex). For example when $n = 8$ we get



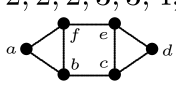
(b) There exists a k-regular graph of order n iff $n > k \geqslant 1$ and at least one of n and k is even. It is necessary that nk be even for the same reason as in (a) (since the sum of the vertex degrees is $nk = 2e$ which must be even). But this condition is also sufficient. If $k = 2r$ is even where $r < \frac{n}{2}$, let Γ be an n-cycle (i.e. circuit of length n as in (a)) and then whenever two vertices v, w of Γ have distance $d(v, w) \in \{2, 3, \ldots, r\}$, add an edge $\{v, w\}$. The resulting graph is $2r$-regular, i.e. k-regular of order n. If instead $k = 2r{+}1$ is odd and n is even, start with the same graph Γ we have just described; and add to this the $\frac{n}{2}$ main diagonals as we did in (a). The resulting graph is $(2r{+}1)$-regular of order n.

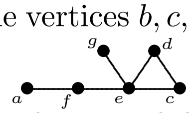
4. Here are a couple of examples of graphs having exactly three automorphisms:



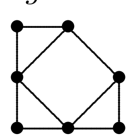
5. (a) If Γ has e edges, then Γ_i has $e - d_i$ edges. The sum of the edges of all graphs in $\mathcal{S}$ is therefore $ne - \sum_i d_i = (n-2)e$, from which we deduce the value of e.

(b) From the value of e which is now known, and the number of edges $e - d_i$ in Γ_i, we deduce the values $e - (e - d_i) = d_i$ for $i = 1, 2, \ldots, n$.

(c) As explained in (a) and (b), we deduce that Γ has 10 edges and degree sequence $2, 2, 2, 3, 3, 4, 4$. The graph Γ is probably easiest to reconstruct from the graph 

since this is missing only one vertex of degree 2. From the degree sequence, we know that the missing vertex (which we call g) must be joined to two of the vertices b, c, e, f. We cannot join g to b and e, otherwise deleting vertex b yields 

, a graph which is not found in original list. By symmetry, the only other possibility is to join g to vertices b and c, yielding

$\Gamma =$ 

This argument *does* uniquely reconstruct Γ. [And of course, you would certainly expect the reconstruction to be unique; otherwise we would have here an exceptionally small counterexample to the famous Graph Reconstruction Conjecture and we would all be famous.]