UNIVERSITY OF WYOMING

Math 3700—Spring 2023

Combinatorics

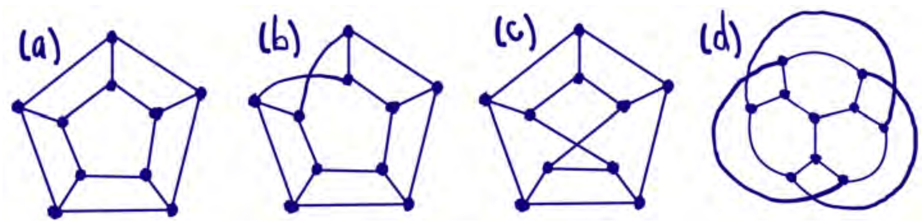
Department of Mathematics

HW1

(Due 5:00 pm, Wednesday, February 15, 2023 *on WyoCourses*)

Instructions: See the syllabus for general instructions for completing homework. Further details are found at the FAQ page linked from the syllabus. Always check your answers wherever feasible. Write clearly, using complete sentences where appropriate, and always using correct notation. All graphs here are simple (undirected, with no loops or multiple edges).

1. *(15 points)* *Graph Isomorphism*: Four graphs are shown, each having 10 vertices and 15 edges.



   (a) Which of these graphs are isomorphic, and which are not? Explain.

   (b) In each case, how many automorphisms does the graph have?

2. *(20 points)* *Infinite Graphs*:

   (a) In class we showed that given a positive integer $n \geqslant 2$, every graph of order n has at least two vertices of the same degree. Does this result hold for infinite graphs? i.e. given a graph with infinitely many vertices, must it have at least two vertices of the same degree?

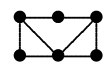
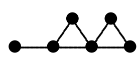
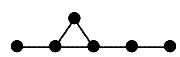
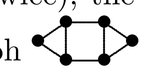
   (b) Does there exist an infinite graph have only one automorphism (the identity)? Explain.

3. *(15 points)* *Regular Graphs*: Recall that a graph is k-regular if each of its vertices has degree k.

   (a) For which positive integers n does there exist a 3-regular graph of order n? Justify your answer.

   (b) State a natural generalization of your answer in (a) for k-regular graphs of order n, and give an informal proof of this statement.

4. *(10 points)* *Graph Automorphisms*: Give an example of a finite graph with exactly 3 automorphisms.

5. *(20 points)* *Graph Reconstruction*: Let Γ be a simple graph on n vertices, and let $\mathcal{S}$ be its *deck*, i.e. the multiset of graphs formed by deleting, in turn, each single vertex of Γ. (So $\mathcal{S}$ consists of n graphs, each with $n-1$ vertices. We may assume Γ has vertex set $\{1, 2, \ldots, n\}$ and Γ_i is formed from Γ by deleting vertex i and all edges from that vertex; thus $\mathcal{S}$ is the list of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. But we are given this list as a multiset of graphs in no prescribed order; we cannot tell which graph came from deleting which vertex.)
   (a) Show that the number of edges in Γ can be uniquely recovered from $\mathcal{S}$.
   (b) Show that from $\mathcal{S}$, we can also uniquely recover the values $d_1, d_2, \ldots, d_n$ where d_i is the degree of vertex i in Γ.
   (c) Suppose that $n = 7$ and that $\mathcal{S}$ consists of the graph  (twice), the graph  (twice), the graph  (twice), and the graph  (once). As pointed out in class, there is a graph Γ having $\mathcal{S}$ as its deck. But is the graph Γ unique up to isomorphism? Explain.