University of Wyoming

Math 3700—Spring 2023

Combinatorics

Department of Mathematics

Final Examination

May, 2023

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Graphs are assumed to be finite and simple (undirected, with no loops or multiple edges) unless clearly specified otherwise. Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value: 100 points (plus 17 bonus points). Time permitted: 120 minutes.

Integer-valued answers having more than a few decimal digits (and which your calculator does not handle) may be expressed using standard notation for factorials $n!$ and falling factorials $P(n,k)$, binomial and multinomial coefficients, Bell numbers B_n, Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$, partition numbers $p(n)$ and $p_k(n)$, etc.

1. *(16 points)* Access to a certain website requires a valid password, defined as a string of characters chosen from an alphabet of 42 symbols: the 26 lower case Roman letters (`a,b,c,...,z`), the 10 digits (`0,1,2,...,9`) and 6 special symbols (`@,$,%,#,&,*`).

   (a) How many passwords of length 8 are there *using only letters*? (e.g. `zildjian` is allowed but `korgms20` is not).

   (b) How many passwords of length 8 are there using at least one letter and at least one special symbol (and any number of digits)? (e.g. `84ib@nez` is allowed but `84ibanez` is not).

   (c) How many passwords of length 8 are there using no character more than once? (e.g. `rickenba` is allowed but `fender#3` is not).

   (d) How many passwords of length 8 are there having no two *consecutive* characters the same? (e.g. `yamaha08` is allowed but `collings` is not).

2. *(21 points)* Let S be the set of all ordinary graphs of order 10 (hence undirected graphs with no loops or multiple edges). The graphs in S are unlabelled (so we have just one graph of each isomorphism type with 10 vertices). It is known that $|S| = 12{,}005{,}168$ (a much smaller number than 2^{45}, the number of labelled graphs with vertex set $[10]$).

   (a) Among all graphs $\Gamma \in S$, what is the maximum possible number of automorphisms of Γ?

   (b) Among all graphs $\Gamma \in S$ which are *bipartite*, what is the maximum possible number of edges of Γ?

   (c) Among all graphs $\Gamma \in S$ which have girth at least 4, what is the maximum possible number of edges of Γ?

   (d) Among all graphs $\Gamma \in S$ which have girth at least 5, what is the maximum possible number of edges of Γ?

   (e) How many of the graphs in S are regular of degree 1?

   (f) How many of the graphs in S are regular of degree 2?

   (g) How many of the graphs in S are regular of degree 7?

3. *(15 points)* I want to partition a supply of coins into envelopes (without any empty envelopes).

   (a) How many ways can I put 12 *identical* coins into *identical* unmarked envelopes?

   (b) How many ways can I put 12 *different* coins into *identical* unmarked envelopes?

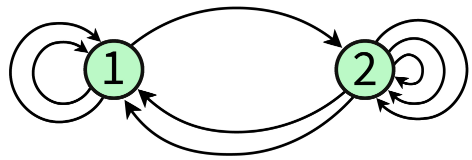
   (c) How many ways can I put 12 *different* coins into *five identical* unmarked envelopes?

   (d) How many ways can I put 12 *identical* coins into *five different* envelopes, marked 'A', 'B', 'C', 'D', 'E'? (and no envelopes empty)

   (e) How many ways can I put 12 *different* coins into *five different* envelopes, marked 'A', 'B', 'C', 'D', 'E'? (and no envelopes empty)

4. *(20 points)* Consider the directed graph Γ with loops and multiple edges, having vertex set $[2] = \{1, 2\}$ and adjacency matrix $A = \left[\begin{smallmatrix} 2 & 1 \\ 2 & 3 \end{smallmatrix}\right]$, as shown.

Γ :



Denote by $w_n = w_n(1, 2)$ the number of walks of length n from vertex 1 to vertex 2. The correct answers to the following are all very simple; if your answers are terribly complicated, please check your work.

(a) Tabulate w_n for $n = 0, 1, 2, 3$.

n	0	1	2	3
w_n				

(b) Give a linear recursive formula of depth 2 for w_n.

(c) Find the generating function $W(x)$ for the sequence w_n. Your answer should be expressed as a rational function (of course in simplified form).

(d) Find the partial fraction decomposition of $W(x)$.

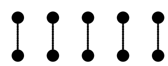
(e) Expand (d) as a power series in x, and read off the exact value of w_n from the coefficients in this series.

(f) Show that w_n satisfies an asymptotic formula $w_n \sim A\alpha^n$ where A, α are explicit constants.

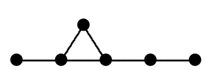
5. *(15 points)* Give explicit answers (not just symbolic expressions) for each of the following. (The largest answer is a 4-digit number.)

   (a) How many 9-letter words can be formed from the word SENSELESS by permuting its letters? (We count all words, i.e. strings of letters, not just English words.)

   (b) How many ways can 9 individual students be divided into four groups of size 4, 3, 1, and 1? (The order of the groups does not matter.)

   (c) How many automorphisms does the graph  have?

   (d) Recall that there are 5 ways to divide a regular pentagon into three triangles using two chords (line segments joining vertices of the pentagon). How many ways can a regular 10-gon be divided into eight triangles using seven chords?

   (e) How many ways can the vertices of the graph  be properly colored using the three colors red, blue, green? (Recall: A proper vertex coloring is one in which no two adjacent vertices have the same color.)

6. *(30 points)* Answer True or False to each of the following statements.

(a) The number of permutations of $[n]$ grows faster than $A\alpha^n$ as $n \to \infty$, for any real constants A and α. ______*(True/False)*

(b) The number of surjections $[2n] \to [n]$ grows faster than the number of injections $[n] \to [2n]$ as $n \to \infty$. ______*(True/False)*

(c) A random graph on n vertices (in which pairs of vertices are joined or not, depending on the outcomes of independent random coin flips) is almost always connected as $n \to \infty$. ______*(True/False)*

(d) If a sequence $a_0, a_1, a_2, \ldots$ satisfies a linear recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$ for some constants $c_1, \ldots, c_r$, then the generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ is a rational function of x. ______*(True/False)*

(e) If a sequence $a_0, a_1, a_2, \ldots$ satisfies a linear recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_r a_{n-r}$ for some constants $c_1, \ldots, c_r$, then the sequence grows at a polynomial rate. ______*(True/False)*

(f) The partition function $p(n)$ has the property that its generating function $\sum_{n=0}^{\infty} p(n) x^n$ is a rational function of x. ______*(True/False)*

(g) If Γ is any graph and $\overline{\Gamma}$ is its complement, then Γ and $\overline{\Gamma}$ have the same group of automorphisms. ______*(True/False)*

(h) There are infinitely many finite graphs (up to isomorphism) having no 10-clique and no 10-coclique. ______*(True/False)*

(i) If A is the adjacency matrix of a k-regular graph Γ of order n, then there exists a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ such that $A\mathbf{v} = k\mathbf{v}$. ______*(True/False)*

(j) If A is the adjacency matrix of a finite ordinary graph Γ, then all roots of the characteristic polynomial of A are real. ______*(True/False)*