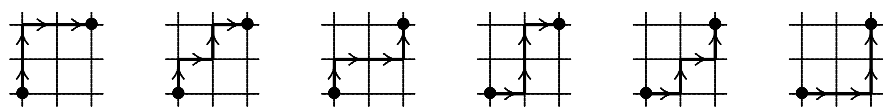
Catalan Numbers

You are walking in a city whose downtown streets run north-south and east-west, forming a grid with all blocks of the same length. We conveniently label positions using (x, y)-coordinates, in such a way that points with integer coordinates are traffic intersections. A shortest path from $(0, 0)$ to $(2, 2)$ (i.e. a path of length 4 blocks) requires walking 4 blocks involves walking two blocks east and two blocks north, in some order; and there are exactly six such shortest paths:

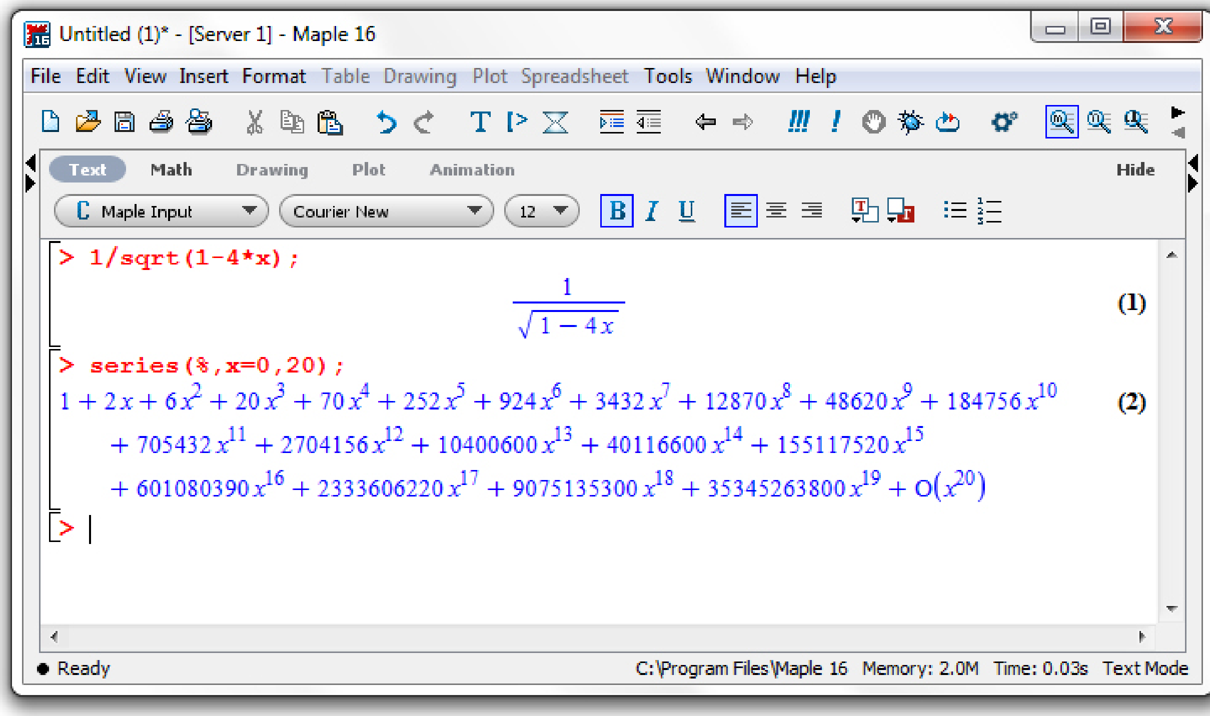


NNEE NENE NEEN ENNE ENEN EENN

More generally, any shortest path from $(0, 0)$ to (n, n) involves walking n blocks north and n blocks east, for a total length of $2n$ blocks; and there are exactly $\binom{2n}{n}$ such shortest paths since this is the number of lists of length $2n$ consisting of n N's and n E's. We remark that the generating function for this value is $\frac{1}{\sqrt{1-4x}}$, as we show using the Binomial Theorem:

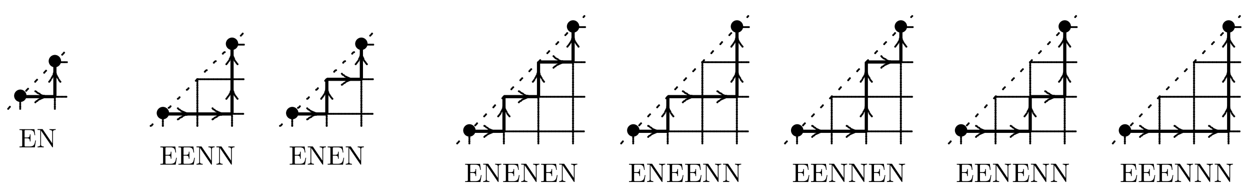
$$\begin{aligned}
\frac{1}{\sqrt{1-4x}} &= (1-4x)^{-1/2} \\
&= \sum_{n\geqslant 0} \binom{-1/2}{n} (-4x)^n \\
&= \sum_{n\geqslant 0} \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) \times \cdots \times (-\frac{2n-1}{2})}{n!} (-2)^n 2^n x^n \\
&= \sum_{n\geqslant 0} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!} 2^n x^n \\
&= \sum_{n\geqslant 0} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{n!} \cdot \frac{2\cdot 4\cdot 6\cdot \cdots \cdot (2n)}{n!} x^n \\
&= \sum_{n\geqslant 0} \frac{(2n)!}{n!\,n!} x^n \\
&= \sum_{n\geqslant 0} \binom{2n}{n} x^n.
\end{aligned}$$

In principle, one could use this generating function to answer our question about the number of shortest paths; for example using the MAPLE session



one can immediately read off the answer for each n. For example for $n = 5$, the coefficient of x^5 reveals that there are $\binom{10}{5} = 252$ shortest paths from $(0, 0)$ to $(5, 5)$. Of course it is probably faster to use the MAPLE command `binomial(10,5);` than to expand the power series as shown, unless for some reason you want to see the answers for an entire range of values of n. But in our next counting problem, the generating function plays a key role.

In this problem, we want to count shortest paths from $(0, 0)$ to (n, n) that do not go above the line $y = x$ (the '*bad side of town*'). The shortest paths for $n = 1, 2, 3$ are as shown:



Let us denote by C_n the number of shortest paths (i.e. paths of length $2n$) fromn $(0, 0)$ to (n, n) that never go above the line $y = x$. This is one of many equivalent definitions available for the sequence of Catalan numbers. If we include the trivial case $C_0 = 1$ (there being only one path of length 0 from the origin to itself), then the first few Catalan numbers are listed in the table

n	0	1	2	3	4	5	6	7	8	9	10
C_n	1	1	2	5	14	42	132	429	1430	4862	16796

We denote the generating function for this sequence by

$$C(x) = \sum_{n\geqslant 0} C_n x^n = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + \cdots.$$

We will first find a simple algebraic formula for $C(x)$, and then expand its power series in order to obtain a closed formula for C_n.

Assume for the moment that $n \geqslant 1$ and consider any shortest path from $(0,0)$ to (n,n) that does not stray above the line $y = x$. Let (k,k) be the first point on our path (*after* the origin) that meets the critical line $y = x$, so that $k \in \{1,2,\ldots,n\}$. This point divides the path into two parts:

(i) the first part of the path, from $(0,0)$ to (k,k), that stays below the critical line $y = x$ except at its endpoints; and
(ii) the second part of the path, from (k,k) to (n,n), that might touch the critical line $y = x$ several times but never goes above it.

The first part (i) of the path must pass through $A = (1,0)$ and $B = (k, k-1)$; and the portion of the path between A and B never goes above the line AB. The number of such paths is clearly C_{k-1}. The second part (ii) of the path has length $2(n-k)$, and there are C_{n-k} such paths. For each value of k, the number of paths satisfying conditions (i) and (ii) is therefore $C_{k-1}C_{n-k}$. Varying k over $\{1,2,\ldots,n\}$, we see that the total number of shortest paths from $(0,0)$ to (n,n) staying below the critical line is

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

whenever $n \geqslant 1$ (and $C_0 = 1$). Multiplying both sides by x^n gives

$$C_n x^n = x\sum_{k=1}^{n}(C_{k-1}x^{k-1})(C_{n-k}x^{n-k})$$

for all $n \geqslant 1$, and then summing over all n, we obtain

$$C(x) = \sum_{n\geqslant 0} C_n x^n = 1 + \sum_{n\geqslant 1} C_n x^n = 1 + x \sum_{\substack{n\geqslant 1\\ 1\leqslant k\leqslant n}} (C_{k-1}x^{k-1})(C_{n-k}x^{n-k}).$$

We change indices of summation using the substitution $(i,j) = (k{-}1,\ n{-}k)$ (equivalently, $(n,k) = (i{+}j{+}1,\ i{+}1)$) to obtain

$$C(x) = 1 + x \sum_{\substack{i\geqslant 0\\ j\geqslant 0}} \big(C_i x^i\big)\big(C_j x^j\big) = 1 + x\Big(\sum_{i\geqslant 0} C_i x^i\Big)\Big(\sum_{j\geqslant 0} C_j x^j\Big) = 1 + xC(x)^2.$$

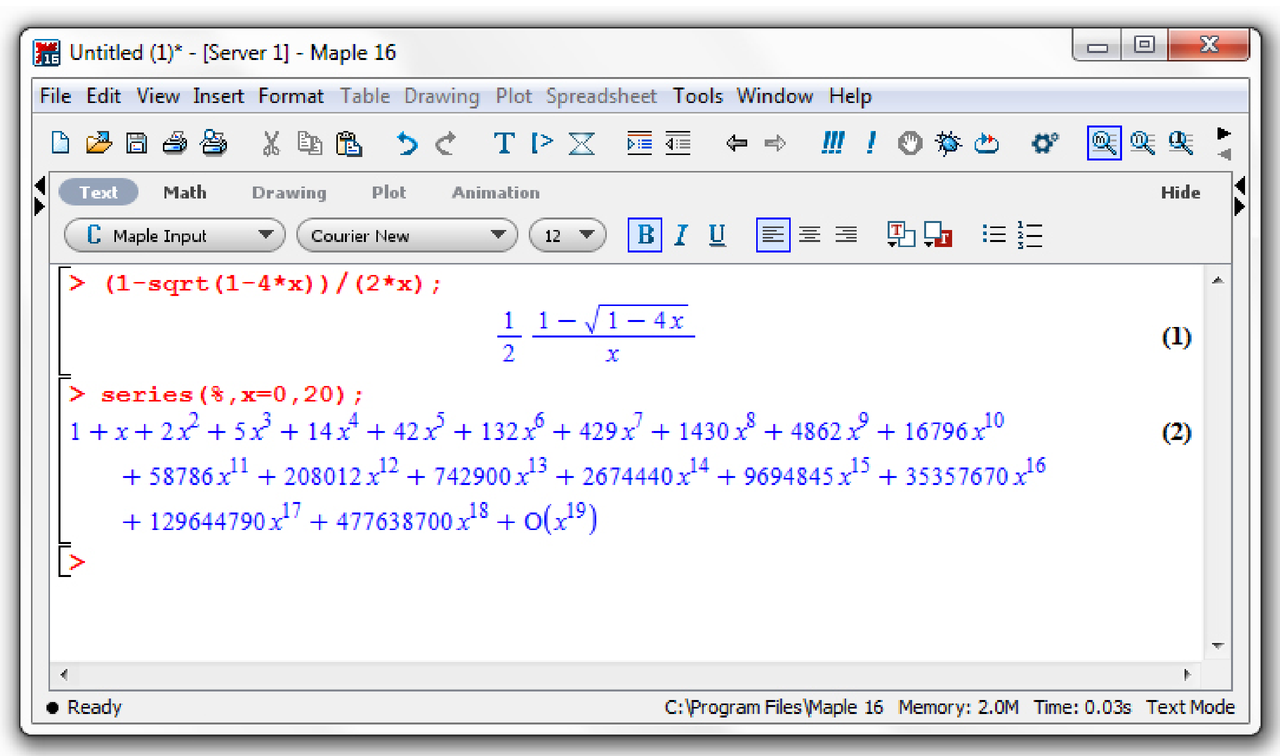
Thus

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm (1 - 2x - 2x^2 - 4x^3 + \cdots)}{2x}\,.$$

We cannot have the '+' sign here as this would yield $C(x) = \frac{1}{x} - 1 - x - 2x^2 - \cdots$ which is impossible for several reasons. (A power series cannot include any terms of negative degree; moreover all coefficients C_n must be $\geqslant 0$.) So we must have the '$-$' sign and

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x} = \frac{1 - (1 - 2x - 2x^2 - 4x^3 + \cdots)}{2x} = 1 + x + 2x^2 + \cdots\,.$$

Before computing a formula for the coefficients (using the Binomial Theorem), let us use MAPLE to check that the power series for this expression agrees with our previous expression, at least for the first few terms:



This agrees perfectly with our previous calculations of the first few Catalan numbers! Now by the Binomial Theorem,

$$\begin{aligned}
C(x) &= \tfrac{1}{2x}\left(1-\sqrt{1-4x}\right)\\
&= \tfrac{1}{2x}\Bigl(1-\sum_{k\geqslant 0}\tbinom{1/2}{k}(-4x)^k\Bigr)\\
&= -\tfrac{1}{2x}\sum_{k\geqslant 1}\tbinom{1/2}{k}(-4x)^k\\
&= -\tfrac{1}{2x}\sum_{k\geqslant 1}\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\times\cdots\times(-\frac{2k-3}{2})}{k!}(-1)^k4^kx^k\\
&= \sum_{k\geqslant 1}\frac{1{\cdot}3{\cdot}5\times\cdots\times(2k-3)}{k!}2^{k-1}x^{k-1}\\
&= \sum_{n\geqslant 0}\frac{1{\cdot}3{\cdot}5\times\cdots\times(2n-1)}{(n+1)!}2^nx^n\\
&= \sum_{n\geqslant 0}\frac{1{\cdot}3{\cdot}5\times\cdots\times(2n-1)}{(n+1)n!}2^nx^n\times\frac{1{\cdot}2{\cdot}3\times\cdots\times n}{n!}\\
&= \sum_{n\geqslant 0}\frac{1{\cdot}3{\cdot}5\times\cdots\times(2n-1)}{(n+1)n!}x^n\times\frac{2{\cdot}4{\cdot}6\times\cdots\times(2n)}{n!}\\
&= \sum_{n\geqslant 0}\frac{(2n)!}{(n+1)n!n!}x^n\\
&= \sum_{n\geqslant 0}\frac{1}{n+1}\binom{2n}{n}x^n.
\end{aligned}$$

Here we can read off directly the coefficient of x^n to obtain

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

for all $n \geqslant 0$. (An interesting consequence of this result is the fact that since C_n is an integer, the binomial coefficient $\binom{2n}{n}$ is always divisible by $n+1$. How would you realize this fact without the derivation above?) It is worth checking this formula at least for the first few of the values $n = 0, 1, 2, 3, \ldots$; again, you will find that the formula agrees with our previously determined values of C_n.