Moore Graphs

In this handout, all graphs are finite and undirected, with no loops or multiple edges. We rely extensively on background in linear algebra (a prerequisite for our course), so pleae review that material before reading ahead. Our course website has links to suitable handouts which cover the material from linear algebra that we require.

A **Moore graph** is a k-regular graph of order n, having diameter 2 and girth 5. (Actually, there is a more general notion of a Moore graph which includes graphs of higher diameter; here we consider exclusively the case of diameter 2.) Students in our course have seen two examples of Moore graphs: the 5-cycle and the Petersen graph. As we shall see, Moore graphs are rare, but there is at least one more which we will reveal later. To study Moore graphs, and to search for the missing examples, requires some background from two areas: linear algebra, and graph theory. We start with the required linear algebra background. While the Spectral Theorem is not necessarily covered in a first undergraduate course in linear algebra, I am providing the necessary additional background here.

The Spectral Theorem for Real Symmetric Matrices

Let A be a real symmetric $n\times n$ matrix. Then A has n real eigenvalues $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$ and a corresponding orthonormal basis of eigenvectors $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n$. In particular, $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is a basis of $\mathbb{R}^n$. The orthonormal property says that the dot product $\mathbf{u}_i^T\mathbf{u}_j=1$ or 0 according as $i=j$ or $i\neq j$. (Here we have represented $\mathbb{R}^n$ as the set of $n\times 1$ *column* vectors with real entries.) There is a more general version of the Spectral Theorem for all normal matrices, including all unitary complex matrices; but here we care only about the case of real symmetric matrices ($A^T=A$ with real entries). The eigenvector condition says, moreover, that $A\mathbf{u}_i=\lambda_i\mathbf{u}_i$ for each $i\in\{1,2,\ldots,n\}$. If we let D be the diagonal matrix with diagonal entries $\lambda_1,\lambda_2,\ldots,\lambda_n$, then $U^{-1}AU=D$, i.e. A is *similar* to the diagonal matrix D. Here U is the $n\times n$ matrix with columns $\mathbf{u}_1,\ldots,\mathbf{u}_n$; and we say that U *diagonalizes* A. Since A is similar to D, its characteristic polynomial is $\det(xI-A)=\det(xI-D)=\prod_{i=1}^n(x-\lambda_i)$. Note that A and D have the same characteristic polynomial, the same eigenvalues, the same determinant, and the same trace (read on...).

The **trace** of a square matrix is the sum of its entries on the main diagonal. If A is an $m\times n$ matrix and B is an $n\times m$ matrix, then AB is an $m\times m$ matrix, while BA is an $n\times n$ matrix. However, both of these matrices have the same trace. Indeed, denoting the trace of a matrix by $\mathrm{Tr}(\cdot)$, and denoting the entries of A and B by a_{ij} and b_{ji}, we have

$$\begin{aligned}\operatorname{Tr}(AB) &= \sum_{i=1}^{m}\big((i,i)\text{-entry of } AB\big) = \sum_{i=1}^{m}\big(\sum_{j=1}^{n} a_{ij}b_{ji}\big)\\ &= \sum_{j=1}^{n}\big(\sum_{i=1}^{m} b_{ji}a_{ij}\big) = \sum_{j=1}^{n}\big((j,j)\text{-entry of } BA\big) = \operatorname{Tr}(BA).\end{aligned}$$

Now assuming $U^{-1}AU = D$ as asserted by the Spectral Theorem, we have

$$\sum_{j=1}^{n}\lambda_j = \operatorname{Tr} D = \operatorname{Tr}(U^{-1}AU) = \operatorname{Tr}(AUU^{-1}) = \operatorname{Tr}(AI) = \operatorname{Tr} A$$

as we have claimed above. This identity will be significant in our study of Moore graphs.

Here are some further explanations which underlie the understanding of the Spectral Theorem, and justifications for its conclusion. We are given a real symmetric matrix A, with characteristic polynomial $f(x) = \det(xI - A)$, a monic real polynomial of degree n. By the Fundamental Theorem of Algebra, we know that $f(x)$ factors into n linear factors $x - \lambda_i$ with $\lambda_i \in \mathbb{C}$. What is not so clear, a priori, is that each $\lambda_i \in \mathbb{R}$. Let's explain why all the eigenvalues of A are in fact real. Let's abbreviate $\lambda = \lambda_i$ and $\mathbf{u} = \mathbf{u}_i$, so that $A\mathbf{u} = \lambda\mathbf{u}$. For any complex matrix matrix X, we denote by X^* the **conjugate transpose** of X (also called the **Hermitian conjugate** of X), i.e. $X^* = \overline{X}^T = \overline{X^T}$. By conjugating both sides of the well-known identity $(XY)^T = Y^TX^T$, we obtain $(XY)^* = Y^*X^*$. Also $(X^*)^* = X$. For any column vector $\mathbf{v} = (a_1, a_2, \ldots, a_n)^T \in \mathbb{C}^n$, we have $\mathbf{v}^*\mathbf{v} = \sum_{i=1}^{n} \overline{a_i}a_i = \sum_{i=1} |a_i|^2 \in [0,\infty)$ (a non-negative *real* number). Of course $A^* = A^T = A$ since A is real symmetric. This yields

$$\lambda|\mathbf{u}|^2 = \mathbf{u}^*(\lambda\mathbf{u}) = \mathbf{u}^*A\mathbf{u} = \mathbf{u}^*A^*\mathbf{u} = (A\mathbf{u})^*\mathbf{u} = (\lambda\mathbf{u})^*\mathbf{u} = \overline{\lambda}\mathbf{u}^*\mathbf{u} = \overline{\lambda}|\mathbf{u}|^2.$$

Since $\mathbf{u}$ is a *nonzero* vector, $|\mathbf{u}|^2$ is a *positive* real number. Thus $\overline{\lambda} = \lambda$, i.e. λ is real.

How is it that $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A (at least in the case A is real symmetric)? First of all, one must realize that eigenvectors corresponding to distinct eigenvalues are necessarily orthogonal; that is, if $\lambda_i \neq \lambda_j$, then the corresponding eigenvectors must satisfy $\mathbf{u}_i^*\mathbf{u}_j = 0$. This follows from

$$\lambda_i\mathbf{u}_i^*\mathbf{u}_j = (\lambda_i\mathbf{u}_i)^*\mathbf{u}_j = (A\mathbf{u}_i)^*\mathbf{u}_j = \mathbf{u}_i^*A^*\mathbf{u}_j = \mathbf{u}_i^*(A\mathbf{u}_j) = \mathbf{u}_i^*(\lambda_j\mathbf{u}_j) = \lambda_j\mathbf{u}_i^*\mathbf{u}_j.$$

Next, one uses Gram-Schmidt to obtain orthonormal bases for the eigenspaces of each of the distinct eigenvalues. We leave the details to your linear algebra textbook.

The Adjacency Matrix of a Graph

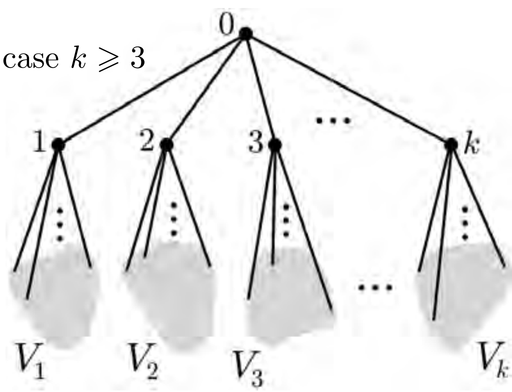
Let Γ be a labelled graph with vertices $1, 2, 3, \ldots, n$. The **adjacency matrix** of Γ is the $n \times n$ matrix whose (i,j)-entry is the number of edges from vertex i to vertex j. (In the case of ordinary graphs, A is a symmetric matrix with entries in $\{0,1\}$, and zeroes on its main diagonal. But here I have defined the adjacency matrix more generally.)

Let $I = I_n$ be the $n \times n$ identity matrix (the $n \times n$ matrix with 1's on its main diagonal, and 0's everywhere else). Let $\mathbf{j} = \mathbf{j}_n$ be the $n \times 1$ column vector, all of whose entries are 1's. Note that Γ is k-regular iff $A\mathbf{j} = k\mathbf{j}$, iff A has $\mathbf{j}$ as an eigenvector with eigenvalue k. An elementary argument (with or without using induction on m) shows that for all $m \geqslant 0$, the (i,j)-entry of A^m is the number of walks of length m from vertex i to vertex j in A. Note that this result holds for all $m \geqslant 0$, including 0 and 1 (since $A^0 = I$ and $A^1 = A$). Moreover, the result holds for *all* finite graphs, including possibly directed graphs, graphs with multiple edges, and graphs with loops.

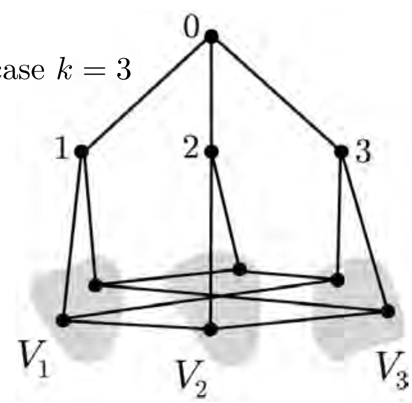
Application to Moore Graphs

Now let Γ be a Moore graph of order n, so Γ is k-regular of diameter 2 and girth 5. Evidently $k \geqslant 2$. Pick any vertex of Γ, which we may label as 0. We choose to label the k neighbors of 0 as $1, 2, 3, \ldots, k$. By the girth condition, Γ contains no triangles or 4-cycles; in particular the vertices $1, 2, \ldots, k$ form a k-coclique. For each $i \in \{1, 2, \ldots, k\}$, let V_i be the set of neighbors of vertex i (*other than* vertex 0), so that $|V_i| = k-1$. Since Γ has no 4-cycles, the sets $V_1, V_2, \ldots, V_k$ are mutually disjoint (i.e. $V_i \cap V_j = \varnothing$ whenever $1 \leqslant i < j \leqslant k$). Since Γ has diameter 2, the set $\{0, 1, 2, \ldots, k\} \sqcup V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ contains *all* vertices of Γ, so $n = 1 + k + k(k-1) = k^2 + 1$. Here is what Γ looks like, both in the general case $k \geqslant 3$, and the special case $k = 3$:

General case $k \geqslant 3$



Special case $k = 3$



Since Γ has no triangles, each V_i is a coclique. So each vertex $x \in V_i$ is joined to $k-1$ vertices in the other V_j's ($j \neq i$). And for all $i \neq j$, each vertex $x \in V_i$ is joined to only one vertex in V_j (again, in order to avoid 4-cycles in Γ. This means that the edges between V_i and V_j form a **matching** (each vertex of V_i is joined to exactly one vertex of V_j). Altogether, the number of edges in Γ is

$$e = k + k(k-1) + \frac{k(k-1)^2}{2} = \frac{(k^2+1)k}{2} = \frac{nk}{2}$$

as required by the formula $nk = 2e$.

We now show that

$(*) \qquad A^2 = kI + (J{-}I{-}A).$

The (i,j)-entry on the left side of $(*)$ is the number of walks of length 2 from vertex i to vertex j. If $i = j$, this number is k since A is k-regular. This agrees with the right side, where we find all k's on the main diagonal. If vertices i and j are adjacent, there are no walks of length 2 from vertex i to vertex j, since Γ has no triangles; thus the left side has (i,j)-entry equal to zero. This agrees with the right hand side, since $J{-}I{-}A$ is the adjacency matrix of the complementary graph $\overline{\Gamma}$. Finally, suppose vertices i and j are not adjacent in Γ. Then the (i,j)-entry of A^2 is 1, since there is a unique walk of length 2 from vertex i to vertex j. (There is at least one such walk, since Γ has diameter 2; and there cannot be more than one such walk, since Γ has no triangles or 4-cycles.) This proves $(*)$.

Now let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of A. Since $A\mathbf{j} = k\mathbf{j}$, one of these eigenvalues is k, with corresponding eigenvector $\mathbf{j}$ (which we may normalize to $\frac{1}{\sqrt{n}}\mathbf{j}$). All the remaining eigenvectors $\mathbf{u}_i$ are orthogonal to $\mathbf{j}$, i.e. $\mathbf{j}^*\mathbf{u}_i = 0$. And the remaining eigenvalues must satisfy $\lambda_i < k$. This follows from $J\mathbf{u}_i = 0$, since

$$\lambda_i^2\mathbf{u}_i = A^2\mathbf{u}_i = k\mathbf{u}_i + (J{-}I{-}A)\mathbf{u}_i = (k{-}1{-}\lambda_i)\mathbf{u}_i,$$

whence $\lambda_i = k{-}1{-}\lambda_i^2 < k$. So in fact the maximum eigenvalue is $\lambda_1 = k$ and it has multiplicity 1. (It is more generally true that for every k-regular graph, the maximum eigenvalue is k; and if the graph is connected, then its multiplicity is 1, and all remaining eigenvalues satisfy $|\lambda_i| < k$. All this is known from the Perron-Frobenius Theorem.)

Thus $\lambda_1 = k > \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ where the remaining eigenvalues λ_i (for $i = 2,3,\ldots,n$) all satisfy $\lambda_i^2 + \lambda_i = k{-}1$. This quadratic equation has only two roots,

$$\lambda = \tfrac{1}{2}\big(-1 + \sqrt{4k{-}3}\big) \quad \text{and} \quad \mu = \tfrac{1}{2}\big(-1 - \sqrt{4k{-}3}\big).$$

Let m be the multiplicity of the eigenvalue λ, so that μ has multiplicity $n{-}m{-}1$. Since the diagonal entries of A are zero, the trace of A is zero; so its diagonal form D also has trace zero. This says that the sum of the eigenvalues is zero:

$$0 = k + m\lambda + (n{-}m{-}1)\mu.$$

Using $n = k^2 + 1$ and the formulas for λ, μ above, this becomes

$(\dagger) \qquad (2m - k^2)\sqrt{4k{-}3} = k(k-2)\,.$

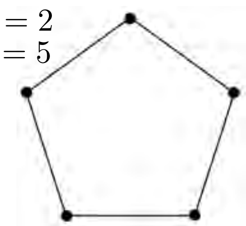
In the case of the 5-cycle ($k = 2$, $n = 5$) we obtain $m = 2$, and the eigvenvalues $\lambda, \mu = \frac{1}{2}\big(-1 \pm \sqrt{5}\big)$ each have multiplicity 2. However when $k > 2$, the preceding equation asserts that the value of $\sqrt{4k{-}3}$ is rational. This requires that $4k{-}3 = d^2$ for some *positive integer* d, and so $(\dagger)$ becomes

$$(d^4 + d^3 + 6d^2 - 2d + 9 - 32m)d = 15.$$

There are only four possible values for a positive integer dividing 15, and we cannot allow $d = 1$ (since $k > 1$), so $d \in \{3, 5, 15\}$. This gives $k \in \{3, 7, 57\}$. The four possibilities for the parameter sets of Moore graphs are summarized in the following table.

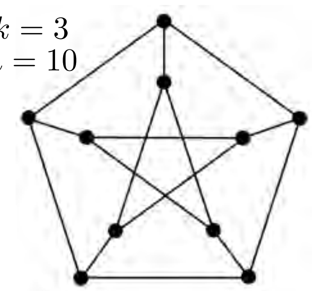
k	n	λ	μ	Γ	$\lvert\mathrm{Aut}\,\Gamma\rvert$
2	5	$\frac{1}{2}\left(-1+\sqrt{5}\right)$ $(2\times)$	$\frac{1}{2}\left(-1-\sqrt{5}\right)$ $(2\times)$	5-cycle	10
3	10	1 $(5\times)$	-2 $(4\times)$	Petersen graph	120
7	50	2 $(28\times)$	-3 $(21\times)$	Hoffman-Singleton graph	252,000
57	3250	7 $(1729\times)$	-8 $(1520\times)$	??	$\leqslant 375$

$k = 2$
$n = 5$



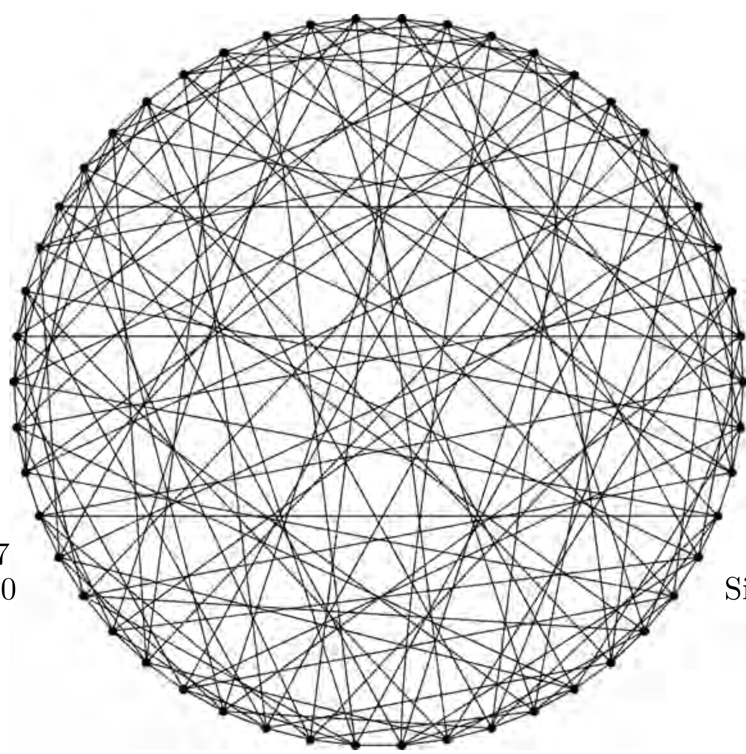
5-cycle

$k = 3$
$n = 10$



Petersen

$k = 7$
$n = 50$



Hoffman-Singleton

It is known that there is a unique Moore graph of each degree $k \in \{2, 3, 7\}$. The existence (and uniqueness) of the Moore graph of degreee 57 remains an open question to this day, although it is known that if the 57-regular Moore graph exists, it cannot have more than 375 automorphisms. In 2020, a manuscript was posted on the arXiv claiming to prove nonexistence of this graph, but the validity of this work has never been verified by the mathematical community.