Solutions to the Test

1. The set of all legal moves of the puzzle is the subgroup $G = \langle \sigma, \tau \rangle \leqslant S_{19}$ where $\sigma = (1, 2, 3, \ldots, 9)$ and $\tau = (9, 10, 11, \ldots, 19)$. Since the generators of G consist of a 9-cycle and an 11-cycle, $G \leqslant A_{19}$. The altered position of the puzzle is obtained by applying the permutation $\pi = (1, 4, 11, 16, 9, 12, 3, 15)(2, 14, 10, 8, 18, 7)(5, 17, 19, 6)$, a product of cycles of length 8, 6 and 4. Since π is an odd permutation, $\pi \notin G$: it is not a legal move.

   Remark: It may be shown that in fact $G = A_{19}$.

2. (a) $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ represents a clcokwise 90° rotation about the origin.

   (b) $\left\langle \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \right\rangle$ represents the group generated by the reflections in the two coordinate axes (e.g. the symmetry group of a rectangle).

   (c) $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ are two examples of elements of infinite order (although almost every element of G has infinite order).

   (d) A concrete example is $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] \in Z(G)$. Any scalar matrix λI with $\lambda \notin \{0, 1\}$ is a valid example.

3. (a) $|S_6| = 6! = 720$.

   (b) S_6 has exactly 144 elements of order 5. They all have the cycle structure (i, j, k, ℓ, m) where $\{i, j, k, \ell, m\}$ is a 5-subset of $[6]$. There are $\binom{6}{5} = 6$ choices of the 5-subset; and if we denote by $i \in [6]$ the smallest label in this 5-subset, the remaining 4 labels j, k, ℓ, m can be listed in any of $4! = 24$ ways. Altogether this gives $6 \cdot 24 = 144$ elements of order 5.

   (c) S_6 has 240 elements of order 6. There are $5! = 120$ six-cycles; and there are $2\binom{6}{3}\binom{3}{2} = 120$ elements with cycle structure $(i, j, k)(\ell, m)$.

4. (a) The subgroup H consisting of matrices of the form $\left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$ is isomorphic to the additive group $\mathbb{R}$. An isomorphism $\phi : \mathbb{R} \to H$ is given simply by $b \mapsto \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]$: this map is bijective, and it satisfies $\phi(b + b') = \phi(b)\phi(b')$.

   (b) The subgroup K consisting of matrices of the form $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$ (with $a \neq 0$) is isomorphic to the multiplicative group $\mathbb{R}^\times$. An isomorphism $\phi : \mathbb{R}^\times \to K$ is given simply by $a \mapsto \left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$: this map is bijective, and it satisfies $\phi(aa') = \phi(a)\phi(a')$.

5. (a) $(1\,3)(1\,2) = (1\,2\,3)$. This will be used in (c).

(b) $(1\,4)(1\,3)(1\,2) = (1\,2\,3\,4)$. This will be used in (d).

(c) $\phi\big((1\,2\,3)\big) = \phi\big((1\,3)\big)\phi\big((1\,2)\big) = (1\,6)(2\,4)(3\,5)(1\,2)(3\,6)(4\,5) = (1\,4\,3)(2\,6\,5)$.

(d) $\phi\big((1\,2\,3\,4)\big) = \phi\big((1\,4)\big)\phi\big((1\,3)\big)\phi\big((1\,2)\big) = (1\,3)(2\,5)(4\,6)(1\,6)(2\,4)(3\,5)(1\,2)(3\,6)(4\,5)$
$= (1\,6\,2\,4)$.

6. (a) T (b) F (c) T (d) T (e) F (f) F (g) T (h) T (i) T (j) F

Here are some remarks and partial explanations for answers in #6:

(a) Since $xy, y^{-1} \in \langle x, y\rangle$, we have $\langle xy, y^{-1}\rangle \leqslant \langle x, y\rangle$. Conversely, both of the elements $x = (xy)(y^{-1})$ and $y = (y^{-1})^{-1}$ are in $\langle xy, y^{-1}\rangle$, so $\langle x, y\rangle \leqslant \langle xy, y^{-1}\rangle$.

(b) A simple counterexample is $x = (1\,2)$ and $y = (1\,3)$ in S_3. There are also easy counterexamples when the group is abelian; e.g. $x = y = (1\,2)$ in S_2.

(c) If x and y commute, then $\langle x, y\rangle = \{x^i y^j : i, j \in \mathbb{Z}\}$ and $(x^i y^j)(x^k y^\ell) = x^{i+k} y^{j+\ell} = (x^k y^\ell)(x^i y^j)$.

(d) If $(xy)^n = e$, then left-multiply by x^{-1} and right-multiply by x to obtain $(yx)^n = e$.

(e) Consider the multiplicative group of all complex roots of unity (elements of finite order in $\mathbb{C}^\times$).

(f) Absolutely not. The symmetry group of a square is a set of transformations. It is not at all the same as the set of things that are being permuted or transformed.

(g) This was proved in class, early in the semester.

(h) Refer to our survey of groups of small order, given in class.

(i) The group S_6 is easily identified as a subgroup of S_7 by extending each $\sigma \in S_6$ to $[7]$ in the trivial way: $\sigma(7) = 7$. (In other wise, identify S_6 as the stabilizer of 7 in S_7.)

(j) Every cyclic group is countable. Also note that the subgroup generated by a nonzero real number $a \in \mathbb{R}$ is $\langle a\rangle = \{na : n \in \mathbb{Z}\}$, which does not contain $a/2$, or πa.