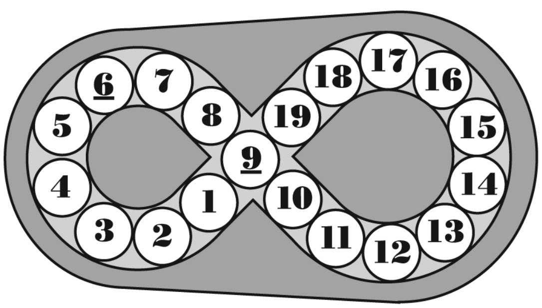
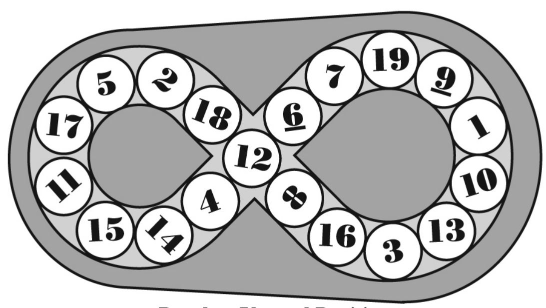
Test—Monday, October 30, 2023

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 65 minutes (8:45–9:50 am). Total value of questions: 100 points (plus 30 bonus points).

1. *(20 points)* A puzzle has 19 round disks numbered 1,2, . . . , 19, which can be moved around a track in the shape of a figure eight, as shown. (One can slide the 9 disks on the left around the left loop, or slide the 11 disks on the right around the right loop; and after every move, one must leave a disk in the middle at the 4-way intersection so that disks are again free to move around either loop.) The objective is to move the disks back to their original factory position, as shown on the left. Explain why, given the position of disks as shown on the right, there is no sequence of moves which will restore the disks to their original factory position.



Puzzle: Original Position



Puzzle: Altered Position

2. *(20 points)* Let $G = GL_2(\mathbb{R})$, the multiplicative group of all invertible 2×2 matrices with real entries.

   (a) Give an explicit example of an element of order 4 in G.

   (b) Find a Klein four-subgroup of G.

   (c) Give an explicit example of an element of infinite order in G.

   (d) Find a non-identity element in G which commutes with every element of G.

3. *(20 points)* Consider the symmetric group S_6.

(a) What is the order of S_6?

(b) How many elements of order 5 does S_6 have?

(c) How many elements of order 6 does S_6 have?

4. *(20 points)* Let G be the multiplicative group consisting of all matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$ where $a, b \in \mathbb{R}$ and $a \neq 0$.

(a) The group G has a subgroup H isomorphic to the additive group of real numbers, $\mathbb{R}$. Find such a subgroup $H \leqslant G$ and an isomorphism $\phi : \mathbb{R} \to H$.

(b) The group G has a subgroup K isomorphic to the multiplicative group of nonzero real numbers, $\mathbb{R}^{\times}$. Find such a subgroup $K \leqslant G$ and an isomorphism $\psi : \mathbb{R}^{\times} \to K$.

5. *(20 points)* There exists an isomorphism $\phi : S_6 \to S_6$ such that

$$\begin{aligned}
\phi\big((1\,2)\big) &= (1\,2)(3\,6)(4\,5)\\
\phi\big((1\,3)\big) &= (1\,6)(2\,4)(3\,5)\\
\phi\big((1\,4)\big) &= (1\,3)(2\,5)(4\,6)\\
\phi\big((1\,5)\big) &= (1\,5)(2\,6)(3\,4)\\
\phi\big((1\,6)\big) &= (1\,4)(2\,3)(5\,6)
\end{aligned}$$

(you should *assume* this). Evaluate each of the following in S_6, in simplified form:

(a) $(1\,3)(1\,2) =$

(b) $(1\,4)(1\,3)(1\,2) =$

(c) $\phi\big((1\,2\,3)\big) =$

(d) $\phi\big((1\,2\,3\,4)\big) =$

6. *(30 points)* Answer TRUE or FALSE to each of the following statements. In (a)–(d), assume that x, y are elements in a multiplicative group G.

(a) The subgroups $\langle x, y\rangle$ and $\langle xy, y^{-1}\rangle$ coincide, i.e. $\langle xy, y^{-1}\rangle = \langle x, y\rangle$. ______*(True/False)*

(b) The order of xy is necessarily the least common multiple of the orders of x and y. ______*(True/False)*

(c) If $xy = yx$, then the subgroup $\langle x, y\rangle$ is necessarily abelian. ______*(True/False)*

(d) The elements xy and yx necessarily have the same order. ______*(True/False)*

(e) If every element of a group G has finite order, then G must have finite order. ______*(True/False)*

(f) The symmetry group of square contains the four vertices and the four sides of the square. ______*(True/False)*

(g) Every subgroup of a cyclic group is cyclic. ______*(True/False)*

(h) Every group of order at most 5 is abelian. ______*(True/False)*

(i) The group S_7 has a subgroup isomorphic to S_6. ______*(True/False)*

(j) The additive group of real numbers is cyclic, generated by the element 1. ______*(True/False)*