Solutions to Sample Exam

1. (a) We have $\sigma \in H$ iff $(6\,7) = \sigma(6\,7)\sigma^{-1} = (\sigma(6), \sigma(7))$ iff $\sigma \in S_5\langle(6\,7)\rangle = \{\alpha, \alpha(6\,7) : \alpha \in S_5\}$ so $|H| = 2|S_5| = 240$. (Alternatively, since $(6\,7)$ has $\binom{7}{2} = 21$ conjugates in G, $|H| = \frac{|G|}{21} = \frac{5040}{21} = 240$.)

   (b) From the description of H above we see that $H \cong S_5 \times \langle(6\,7)\rangle \cong S_5 \times C_2$ where C_2 is a cyclic group of order 2. The isomorphism $H \to S_5 \times \langle(6\,7)\rangle$ is given by $\alpha \mapsto \big(\alpha, ()\big)$; $\alpha(6\,7) \mapsto \big(\alpha, (6\,7)\big)$.

2. Note that $T_\theta^{-1} = T_\theta$, $R_\alpha R_\beta = R_{\alpha+\beta}$ and $R_\theta^{-1} = R_{-\theta}$. For all θ, we have $T_\theta T_0 = R_{2\theta}$ as is seen either by considering the action of both sides on a pair of vectors (such as the unit vector on the positive x-axis, and a unit vector on the axis of T_θ) or using matrices:

$$T_\theta T_0 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} = R_{2\theta}\,.$$

In particular, $R_\theta = T_{\theta/2} T_0$ which gives (a). Also, solving for T_θ gives

$$T_\theta = R_{2\theta} T_0 = T_0 R_{-2\theta}\,.$$

Using these identities,

$$R_\theta T_\alpha R_\theta^{-1} = R_\theta (R_{2\alpha} T_0) R_{-\theta} = R_{\theta+2\alpha} T_0 R_{-\theta} = T_0 R_{-\theta-2\alpha} R_{-\theta} = T_0 R_{-2\theta-2\alpha} = T_{\alpha+\theta}\,.$$

So to solve (b), take $\theta = \beta - \alpha$.

All these solutions can be checked directly using 2×2 matrix representations for isometries. Alternatively, if we identify D with the disk $|z| \leqslant 1$ in the complex plane, then

$$T_0(z) = \overline{z}; \quad R_\theta(z) = e^{i\theta} z; \quad T_\theta(z) = e^{2i\theta}\overline{z}$$

so

$$T_{\theta/2}(T_0(z)) = e^{i\theta}\overline{\overline{z}} = e^{i\theta} z = R_\theta(z);$$
$$R_\theta(T_\alpha(R_{-\theta}(z))) = e^{i\theta} e^{2i\alpha}\overline{e^{-i\theta} z} = e^{2i(\alpha+\theta)}\overline{z} = T_{\alpha+\theta}(z).$$

3. For all $A, B \in G$ we have

$$\theta(AB) = \det(AB)AB = \det(A)\det(B)AB = \big(\det(A)A\big)\big(\det(B)B\big) = \theta(A)\theta(B)$$

so $\theta : G \to G$ is a homomorphism. An inverse for θ is the map $\psi : G \to G$ given by $\psi(A) = (\det A)^{-1/3}A$. To verify this, note that if $\delta = \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then

$$\psi\big(\theta\big(\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)\big)\big) = \psi\big(\big(\begin{smallmatrix} \delta a & \delta b \\ \delta c & \delta d \end{smallmatrix}\big)\big) = (\delta^3)^{-1/3}\big(\begin{smallmatrix} \delta a & \delta b \\ \delta c & \delta d \end{smallmatrix}\big) = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)$$

so $\psi \circ \theta$ is the identity map $G \to G$; similarly, $\theta \circ \psi : G \to G$ is the identity map. Thus θ is an automorphism of G. (It is crucial that these matrices have real entries, since we used the fact that cube roots of real numbers are well defined.)

4. (a) Yes, $H \cap K \trianglelefteq H$. Suppose $k \in H \cap K$ and $h \in H$; then $hkh^{-1} \in H$ by closure of H, and $hkh^{-1} \in K$ since $K \trianglelefteq G$; so $hkh^{-1} \in H \cap K$.

   (b,c) No, $H \cap K$ need not be normal in K or in G; for example consider $G = S_4$, $H = S_3$, $K = A_4$, $H \cap K = A_3 = \langle(123)\rangle$. Conjugates of (123) in A_4, or in G, give other three-cycles such as $(12)(34)\cdot(123)\cdot(12)(34) = (142) \notin \langle(123)\rangle$.

5. (a) Yes, G has a subgroup

$$\left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} : a,b,c,d \in \mathbb{R},\ ad - bc \neq 0 \right\} \cong GL_2(\mathbb{R})$$

   where the isomorphism is given by

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

   (b) Yes; for example $\langle R_{2\pi/5}\rangle$ where the rotation by angle θ about the z-axis is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (c) Yes; for example consider the group of all rotations R_θ about the z-axis, as found in (b). Alternatively, consider all matrices of the form

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b \in \mathbb{R}.$$

6. The stabilizer of $\mathbf{u} = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ is $G_\mathbf{u} = \left\{\left[\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right] : a, c \in \mathbb{F}_5,\ a \neq 0\right\}$, the same as the stabilizer of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Note that $|G_\mathbf{u}| = 20$.

7. (a) $G = A_6$ has $\binom{6}{4}3! = 90$ elements of order 4; these are the elements having the same cycle structure as $\sigma = (1\,2\,3\,4)(5\,6)$. In fact, $C_G(\sigma) = \langle\sigma\rangle$ is cyclic of order 4 and the number of conjugates of σ is $[G : C_G(\sigma)] = \frac{360}{4} = 90$.

   (b) $G = A_6$ has $\binom{6}{5}4! = 144$ elements of order 5. These are partitioned into two conjugacy classes of size 72, represented by $\sigma = (1\,2\,3\,4\,5)$ and $\sigma' = (1\,2\,3\,5\,4)$. (Any permutation in S_6 conjugating σ to σ' is necessarily an odd permutation.) In fact, $C_G(\sigma) = \langle\sigma\rangle$ is cyclic of order 5 and the number of conjugates of σ is $[G : C_G(\sigma)] = \frac{360}{5} = 72$.

8. (a) T (b) T (c) T (d) T (e) F (f) T (g) T (h) F (i) T (j) F

Comments in #8:

(a) This is the statement of Cayley's Representation Theorem.

(b) Consider a nontrivial element $g \in G$ has order $n > 1$, and let p be a prime divisor of n; then $|g^{n/p}| = p$.

(c) Conjugation by y maps $(xy)^6 \mapsto (yx)^6$. Since this is an inner automorphism, $|(xy)^6| = |(yx)^6|$. Alternatively, note that $(xyxyxyxyx)y = 1$ iff $y(xyxyxyxyx) = 1$.

(d) This standard result was proved in class, as follows. If θ is one-to-one and $\theta(g) = 1 = \theta(1)$, then $g = 1$, so $\ker\theta = \{1\}$. Conversely if $\ker\theta = \{1\}$ and $\theta(x) = \theta(y)$, then $\theta(xy^{-1}) = \theta(x)\theta(y)^{-1} = 1$ so $xy^{-1} \in \ker\theta = \{1\}$, whence $xy^{-1} = 1$, i.e. $x = y$.

(e) For example a Klein 4-group $K = \{1, a, b, c\}$ has $\mathrm{Aut}\,K \cong S_3$ permuting a, b, c in all $3! = 6$ possible ways, and S_3 is nonabelian. More generally if V is the additive group of a vector space of dimension $\geqslant 2$, then $\mathrm{Aut}\,V$ contains invertible linear transformations which in general do not commute. (This is a well known property of invertible matrices.)

(f) Use the fact that the subgroups G_x and G_y are conjugate. Or use $|G_x| = \frac{|G|}{|X|} = |G_y|$.

(g) Consider $\langle(12)\rangle\langle(13)\rangle = \{(), (12), (13), (132)\}$ in S_3.

(h) Consider g to be the nonidentity element in a group of order 2; or an element of order 2 in a Klein 4-group; or a generator of a cyclic group of order 4; or any odd permutation in S_n.

(i) There exist integers r, s such that $rk + sn = 1$. Then $\theta_r(\theta_k(x)) = x^{rk} = x^{1-sn} = (x^n)^{-s}x = 1x = x$ so $\theta_r \circ \theta_k = \theta_k \circ \theta_r$ is the identity map $G \to G$. (Note: θ_k is not usually a homomorphism, unless G is abelian.)

(j) In class we gave an example of two nonisomorphic groups of order 27, both having 26 elements of order 3.