Solutions to Sample Test

October, 2023

1. One solution is $G = \{(),\ (1234),\ (13)(24),\ (1432),\ (13),\ (24),\ (12)(34),\ (14)(23)\}$ obtained by considering the permutation action of G on the four vertices of a square, which we have chosen to label as 1,2,3,4 in counter-clockwise order. The isomorphism of our group $G \leqslant S_4$ with the symmetry group of the square (call it D_8 for the dihedral group of order 8) is easily established: every symmetry of the square $g \in D_8$ permutes the four vertices, yielding a permutation $\phi(g) \in S_4$; and the map $\phi : D_8 \to S_4$ is one-to-one since every symmetry of the square is uniquely reconstructible from its action on the four vertices. Note that G is exactly the image of ϕ. The fact that $\phi(g \circ h) = \phi(g) \circ \phi(h)$ for all $g, h \in D_8$ is clear: the permutation $\phi(g \circ h)$ induced by $g \circ h$ on the four vertices is the composite of the permutation $\phi(h)$ followed by the permutation $\phi(g)$.

   In fact S_4 has exactly three subgroups isomorphic to D_8: in addition to the example above, there are subgroups

   $$\{(),\ (1324),\ (12)(34),\ (1423),\ (12),\ (34),\ (13)(24),\ (14)(23)\}$$

   and

   $$\{(),\ (1243),\ (14)(23),\ (1342),\ (14),\ (23),\ (12)(34),\ (13)(24)\}$$

   arising from alternative labelings of the four vertices of the square (1,3,2,4 or 1,2,4,3 in counter-clockwise order, respectively).

2. All elements of S have determinant ± 1; so every product of elements of S has determinant ± 1. No element of $GL_2(\mathbb{R})$ having any other determinant than 1 or -1, can be expressed as a product of elements of S. (In fact $\{A \in GL_2(\mathbb{R}) : \det A = \pm 1\}$ is a subgroup of $GL_2(\mathbb{R})$; and it may be shown that $\langle S \rangle = G$, i.e. the matrices expressible as products of elements of S are precisely the elements of G.)

3. (a) Almost nothing can be said about $|gh|$ in general. Clearly $|gh| \neq 1$; otherwise g and h would be inverses of each other, which is not possible since they do not have the same order. But from the information given, $|gh|$ could be any integer $\geqslant 2$, or even infinite. Consider these examples in S_8:

   $|(123)(45678)| = 15$

   $|(123)(34567)| = |(1234567)| = 7$

   $|(123)(23456)| = |(12)(3456)| = 4$

   $|(123)(12345)| = |(13452)| = 5$

   $|(123)(14352)| = |(14)(35)| = 2$

   $|(123)(13245)| = |(245)| = 3$

(b) If g and h commute then $|gh| = 15$. To see this, note that $(gh)^k = g^k h^k$ and in particular $(gh)^{15} = g^{15}h^{15} = 1$; but $g^k h^k \neq 1$ for $k \in \{1, 2, 3, \ldots, 14\}$ since $g^k \neq h^{-k}$ (the left side has order 1 or 3, but the right side has order 1 or 5).

4. In S_3 we have $\langle(12)\rangle\langle(13)\rangle = \{(),\ (12)\}\{(),\ (13)\} = \{(),\ (12),\ (13),\ (132)\}$ which is not a subgroup; it is not closed since it does not contain the product $(13)(12) = (123)$.

5. (a)T (b)T (c)F (d)T (e)T (f)T (g)T (h)F (i)T (j)T

Here are some remarks and partial explanations for answers in #5:

(a) For every positive integer n, there is a cyclic group of order n.

(b) Every group isomorphism is in particular a bijection.

(c) There are only finitely many groups of order n up to isomorphism, since there are only finitely many ways to complete a Cayley table with the symbols $1, 2, \ldots, n$.

(d) If G is a finite group, and $g \in G$, then the elements $1, g, g^2, g^3, \ldots \in G$ cannot all be distinct; so $g^i = g^j$ for some $i < j$, in which case $g^{j-i} = 1$ and $|g| \leqslant j - i$.

(e) Here are two objects, each with 180° degree rotational symmetry, but no reflective symmetry:

(f) The multiplicative group $\mathbb{C}^\times$ has an element 2 of infinite order; and for each positive integer n, the element $e^{2\pi i/n}$ has order n.

(g) The map $x \mapsto 2x$ is an isomorphism from the additive group of integers $\mathbb{Z}$ to the additive group of even integers $2\mathbb{Z} = \{2k : k \in \mathbb{Z}\}$.

(h) In the group G of transformations of the Euclidean plane, take g to be a reflection in the y-axis, and h to be a reflection in the line $x = 1$; then gh is a translation 2 units to the right; and in particular, h has infinite order.

(i) It is easy to verify the group axioms for the second binary operation. Note that for $g \in G$, the inverse of g with respect to '$\circ$' is the same as the inverse of g with respect to '$*$'; and denoting this inverse by g^{-1}, the map $g \mapsto g^{-1}$ is an isomorphism from one grou to the other.

(j) It is a simple matter to rename the elements of G as $1, 2, 3, \ldots, n$.