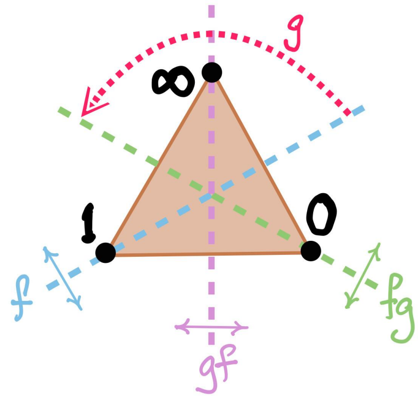
Solutions to HW4

1. The group $G = \langle f, g\rangle = \{\iota, f, g, g^2, fg, gf\}$ has order $|G| = 6$, using right-to-left composition $(gf)(x) = g(f(x)) = 1 - x$ and denoting the identity by $\iota(x) = x$. We have
   - one element ι of order 1;
   - three elements of order 2:
     $f(x) = \frac{1}{x}$,
     $(gf)(x) = 1 - x$,
     $(fg)(x) = \frac{x}{x-1}$; and
   - two elements of order 3:
     $g(x) = \frac{x-1}{x}$,
     $g^2(x) = \frac{1}{1-x}$.



   Based on the number of elements of each order, or the fact that $fg \neq gf$, we know that $G \cong S_3$, the unique nonabelian group of order 6. For an explicit isomorphism, note that G gives all six permutations of $\{0, 1, \infty\}$. In cycle notation, $\iota = ()$; $f = (0, \infty)$; $fg = (1, \infty)$; $gf = (0, 1)$; $g = (0, \infty, 1)$; $g^2 = (0, 1, \infty)$. We may interpret the elements of G as symmetries of an equilateral triangle with vertices $0, 1, \infty$ as shown above.

2. We may choose $\tau = (354)$; but composing with automorphisms gives altogether 8 different choices of isomorphism $\Gamma_2 \to \Gamma_9$. Note that $\mathrm{Aut}\,\Gamma_2 = \langle(14), (1245)\rangle$ and $\tau\langle(14), (1245)\rangle\tau^{-1} = \langle(13), (1234)\rangle = \mathrm{Aut}\,\Gamma_9$ as required.

3. Since Γ_{11} is a 6-cycle, $\mathrm{Aut}\,\Gamma_{11} = \langle(126354), (15)(23)\rangle$ is a dihedral group of order 12.

4. $\mathrm{Aut}\,\Gamma_{12} = \langle(136425), (12)(34)\rangle$ is a dihedral group of order 12. The complementary graph Γ'_{12} is a 6-cycle with vertices 136425 (in cyclic order).

5. G has
   1 element of order 1;
   7 elements of order 2;
   24 elements of order 4; and
   32 elements of order 8.

   We can take G to be the set of all triples of the form (u^i, v^j, w^k) where $|u| = 24$, $|v| = 4$, $|w| = 8$; also $i \in \{0, 1\}$, $j \in \{0, 1, 2, 3\}$, $k \in \{0, 1, 2, 3, 4, 5, 6, 7\}$. Every element $g \in G$ satisfies $g^8 = 1$, so $|g| \in \{1, 2, 4, 8\}$. The 32 elements with k odd have order 8.

The eight elements with $i \in \{0,1\}$, $j \in \{0,2\}$, $k \in \{0,4\}$ form a subgroup isomorphic to $C_2 \times C_2 \times C_2$ with 7 elements of order 2 and 1 element of order 1. All the 24 remaining elements of G have order 4.

6. (a) Let $H = \{(\alpha, \phi(\alpha)) : \alpha \in S_5\} \subset G$. The obvious bijection $f : S_5 \to H$ given by $f(\alpha) = (\alpha, \phi(\alpha))$ satisfies
$$f(\alpha\beta) = (\alpha\beta, \phi(\alpha\beta)) = (\alpha, \phi(\alpha))(\beta, \phi(\beta)) = f(\alpha)f(\beta),$$
so H is a subgroup and f is an isomorphism. Here you get a 'free pass' in verifying that H is a subgroup (closure, identity, and inverses follow for free) because f is an isomorphism from a known group S_5 to a subset $H \subset G$ for another known group $G = S_5 \times S_5$.

   (b) We really only need $\phi : S_5 \to S_5$ to be a homomorphism; the proof in (a) gives us a subgroup $H < G$ (which depends on the choice of ϕ). But ϕ definitely needs to be a homomorphism; an arbigrary function $\phi : S_5 \to S_5$ will not work.

   (c) There are exactly 172 subgroups of G isomorphic to S_5. I would not expect you to be able to justify why there are *only* 172 such subgroups, but if you can see why there are *at least* 172 such subgroups H, I will give you full credit. We include

   - The two subgroups $S_5 \times 1$ and $1 \times S_5$ described in the HW4 handout.
   - The 120 subgroups $\{(\alpha, \phi(\alpha)) : \alpha \in S_5\}$ where $\phi \in \operatorname{Aut} S_5$. As mentioned in class, all such automorphisms are inner, i.e. they have the form $\phi_\tau(\alpha) = \tau\alpha\tau^{-1}$. The map $S_5 \to \operatorname{Aut} S_5$, $\tau \mapsto \phi_\tau$ is an isomorphism.
   - There are 25 elements $\tau \in S_5$ of order 2 in S_5. Each gives rise to a homomorphism $\phi : S_5 \to \langle\tau\rangle$ where $\phi(\alpha) = ()$ or τ according as α is even or odd. The resulting pairs $(\alpha, \phi(\alpha))$ produce 25 distinct subgroups of G isomorphic to S_5.
   - Repeat the previous 25 examples using the pairs $(\phi(\alpha), \alpha)$ instead of $(\alpha, \phi(\alpha))$. This gives another 25 distinct subgroups isomorphic to S_5.

7. The set $[10] = \{1, 2, \ldots, 10\}$ has $\binom{10}{5} = 252$ subsets of size 5, and $\frac{1}{2}\binom{10}{5} = 126$ pairs of disjoint subsets of size 5, such as $\{\{1,3,4,6,9\}, \{2,5,7,8,10\}\}$. We write $[10] = A \sqcup A'$ to say that $\{A, A'\}$ is a partition of $[10]$ into two disjoint subsets of size 5. Corresponding to any such partition, we have a subgroup of S_{10} isomorphic to G, consisting of all $\sigma \in S_{10}$ preserving both A and A'. We obtain 126 subgroups isomorphic to G in this way. These subgroups are all conjugate in S_{10}. For example, consider the partitions $[10] = \{1,2,3,4,5\} \sqcup \{6,7,8,9,10\}$ and $\{1,3,4,6,9\} \sqcup \{2,5,7,8,10\}$. The permutation $\tau = (26)(59)$ maps the first partition to the second, so it conjugates one subgroup (isomorphic to G) to the other.

   Are these 126 subgroups of S_{10} the *only* ones isomorphic to G? Yes, but I would not expect you to be able to prove that fact given what you know.