Solutions to HW3

1. Every value obtained as a product of all elements of S_3 must be odd, since we are multiplying together three even and three odd permutations. Conversely, it is easy to check that each transposition is expressible as a product of all the elements of S_3:

$$\begin{aligned}(12) &= ()(123)(132)(13)(12)(23),\\ (13) &= ()(123)(132)(12)(13)(23),\\ (23) &= ()(123)(132)(12)(23)(13).\end{aligned}$$

   Thus the elements of S_3 expressible as a product of all the elements of S_3 are *exactly* (12), (13) and (23).

   After learning about conjugation, we recognize that a shorter argument suffices: Taking the product of the elements of S_3 in any order, gives one of the three transpositions. Then we can conjugate this formula to get any of the three transpositions as a product of all the elements of S_3.

2. The product of all the elements of $C_n = \{1, 2, \ldots, g^{n-1}\}$ is 1 if n is odd, or $g^{n/2}$ (the unique element of order 2) if n is even. Note that in the product of all elements of C_n, each element $x = g^i$ cancels with its inverse $x^{-1} = g^{-i}$, unless $x = x^{-1}$. An element satisfies $x = x^{-1}$ iff $x^2 = 1$, iff $|x| = 1$ or 2. If n is odd, then C_n has no element of order 2, so the product of all elements of C_n is 1. If n is even, then after cancellation, we are left with $1{\cdot}g^{n/2} = g^{n/2}$, the unique element of order 2.

3. There are exactly 10 homomorphisms $\phi : S_3 \to S_3$. Each such homomorphism is uniquely determined by the two values $\phi((12)), \phi((13)) \in \{(), (12), (13), (23)\}$ since $S_3 = \langle (12), (13)\rangle$. Here are all possible cases:
   - (i) If $\phi((12)) = \phi((13)) = ()$, then we have the trivial homomorphism $\phi(\sigma) = ()$ for all $\sigma \in S_3$. (Just one possibility for ϕ in this case.)
   - (ii) If $\phi((12))$ and $\phi((23))$ are distinct transpositions, then we have exactly six possiblities for ϕ; and these are exactly the six inner automorphisms $\phi = \psi_\tau$ for $\tau \in S_3$, namely conjugation $\phi(\sigma) = \psi_\tau(\sigma) = \tau\sigma\tau^{-1}$. (Six possibilities for ϕ in this case.)
   - (iii) If $\phi((12)) = \phi((13)) = \tau \in \{(12), (13), (23)\}$ is the same transposition, then $\phi(\sigma) = ()$ or τ according as σ is even or odd. This is a homomorphism, amounting to the sign homomorphism $S_3 \to \{\pm 1\}$, but with ± 1 renamed as powers of the transposition τ. (Three possibilities for ϕ in this case.)

(iv) If one of $\phi((12)), \phi((13))$ is $()$ and the other is a transposition, we have no possibilities for ϕ. For example if $\phi((12)) = ()$, then $\phi((13)) = \phi((23)(12)(23)) = \phi((23))\phi((12))\phi((23))^{-1} = ()$, a contradiction. (Zero possibilities for ϕ in this case.)

4. (a) Since $\phi(1_G) = \phi_G(1_G 1_G) = \phi(1_G)\phi(1_G)$, we can left-multiply both sides by $\phi(1_G)^{-1}$ to obtain $\phi(1_G) = 1_H$.

   (b) Since both ϕ and ψ are bijective, so is $\psi \circ \phi$. Moreover, $\psi \circ \phi$ is a homomorphism since $(\psi \circ \phi)(ab) = \psi(\phi(ab)) = \psi(\phi(b)\phi(a)) = \psi(\phi(a))\psi(\phi(b)) = (\psi \circ \phi)(a)(\psi \circ \phi)(b)$. So $\psi \circ \phi : G \to K$ is an isomorphism.

5. The element $B = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ works, as does $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Any matrix B with the required property must satisfy $A^{-T}B = BA$ for all $A \in G$, giving a system of four linear equations in the four unknown entries of B. Only three of these linear equations are independent, so we obtain solutions of the form $B = c\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ for $c \in \mathbb{R}$. Finally, using the requirement that $\det B = 1$ gives $c = \pm 1$, and so the only two solutions are the ones we gave above.

   Some students will try to directly use the relation $A^{-T} = BAB^{-1}$ in its original unsimplified form, but this leads to a system of quadratic equations in the unknown entries of B, requiring a little more work. This is fine if you see one of the solutions by inspection, inasmuch as you were not really asked to find all possibilities for B. But the simplification described above is helpful and worth remembering.

6. The smallest example is $[2] \in GL_1(\mathbb{F}_3)$ which is a transposition of the vectors, fixing (0), and interchanging (1) with (2). A small example with $n > 1$ is $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in GL_2(\mathbb{F}_2)$ which fixes each of the vectors $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, while transposing the remaining two vectors as $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \leftrightarrow \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Another example is $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right] \in GL_2(\mathbb{F}_3)$, which permutes the nine vectors of $\mathbb{F}_3^2$; it fixes each of the three vectors $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ for $x \in \mathbb{F}_3$, and interchanges the six remaining vectors in pairs as $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \leftrightarrow \left[\begin{smallmatrix} x \\ 2 \end{smallmatrix}\right]$. This permutation is therefore a product of three disjoint transpositions, which is odd.

   This is not a hard problem. The main point of this is to learn to think of groups concretely via their actions on sets (in this case the general linear group acting on vectors) and to think about properties of the group via such actions.