Solutions to HW1

1. (a) G is nonabelian of order 20. The prism has 10 corners; and for any two corners, G has exactly two isometries taking one to the other (one of which preserves orientation, and the other reverses orientation). There is one 'horizontal' plane of symmetry; also five vertical planes of symmetry, in planes spaced 36° apart. The reflection in the horizontal plane of symmetry commutes with any of the reflections in vertical planes of symmetry; but two of the reflections in vertical planes of symmetry do not commute.

   (b) G is abelian of order 2 (a cyclic group generated by the reflection in the horizontal line of symmetry).

   (c) G is abelian of order 2 (a cyclic group generated by the half-turn, i.e. 180° rotation, about the center of the O.

   (d) G is infinite abelian. It is generated by the translation one unit to the right (a unit being the distance between the centers of two adjacent E's; and the reflection in the horizontal axis of symmetry. The two generators commute with each other.

   (e) G is abelian of order 1. It is the trivial group, since the string of letters has no nontrivial symmetries.

   (f) G is infinite nonabelian. Let ℓ_1 be the horizontal line of symmetry; for ℓ_2 take the vertical line of symmetry half-way between two adjacent H's; and let ℓ_3 be the vertical line of symmetry through the middle of the H immediately to the right of ℓ_2. The reflections R_1, R_2, R_3 in the axes ℓ_1, ℓ_2, ℓ_3 respectively, are generators for G. Note that R_2R_3 is a shift one unit to the right, whereas $R_3R_2 = (R_2R_3)^{-1}$ is a shift one unit to the left, so G is nonabelian.

   (g) G is abelian of order 8, generated by the reflections in the three planes of symmetry. All seven non-identity elements of G have order 2; including the three half-turns (180° rotations) about the three axes of symmetry; and the inversion in the center (mapping each vector to its negative, assuming we place the origin of our coordinate system at the center of the brick).

2. (a) $G = \{\,(), (1324)(5768), (1423)(5867), (1526)(3847), (1625)(3748), (1728)(3546), (1827)(3645), (12)(34)(56)(78)\,\}$.

   (b) G has 1 element of order 1, 6 elements of order 4, and 1 element of order 2. In particular, G is not dihedral.

(c) No, G is nonabelian; two elements of order 4 do not commute, except when one is the inverse of the other. This is in fact the quaternion group of order 8. (There are two nonabelian groups of order 8: the dihedral group and the quaternion group of order 8.)

3. An example is (123). Note that $(123)\cdot(14)(25)(36) = (142536)$ whereas $(14)(25)(36)\cdot(123) = (152634)$.

4. (a) $\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\-1&-1\end{bmatrix}, \begin{bmatrix}-1&-1\\1&0\end{bmatrix}, \begin{bmatrix}-1&-1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\-1&-1\end{bmatrix}\right\}$

   (b) G has 1 element of order 1, 2 elements of order 3, and 3 elements of order 2. (These elements have trace 2, -1, and 0 respectively.)

   (c) No, G is nonabelian; no two of the elements of order 2 (i.e. trace zero) commute. In fact, $G \cong S_3$.