Solutions to the Final Examination

December, 2023

As usual, I have added remarks and tried to give two or three different solutions whenever reasonable. This, however, makes the solutions appear much longer...

1. (a) σ has $2\binom{6}{3} = 40$ conjugates in G. For the first 3-cycle (i,j,k), choose $\{i,j,k\}$ in $\binom{6}{3} = 20$ possible ways; then multiply by 2 to get 40 different 3-cycles (i,j,k). There are the same number of permutations having cycle structure $(i,j,k)(\ell,m,n)$ in G. (Given the first 3-cycle (i,j,k), the remaining three points ℓ,m,n can be cycled in either of two directions, giving $(i,j,k)(\ell,m,n)$ and $(i,j,k)(\ell,n,m)$ with the desired cycle structure; and this gives $40 \cdot 2 = 80$. But then divide by 2 again to avoid overcounting, because $(i,j,k)(\ell,m,n) = (\ell,m,n)(i,j,k)$: either of the two 3-cycles might have been chosen as the 'first' 3-cycle.)

   It may be easier to answer (b) first, and then to use the formula for the number of conjugates: $[G : C_G(\sigma)] = \frac{720}{18} = 40$.

   A third explanation for the number 40 is that it is exactly the same as the number of 3-cycles in G, which is also $2\binom{6}{3} = 40$. As explained in class, G has two conjugacy classes of elements of order 3; and these two conjugacy classes have the same size, due the outer automorphism (shown on the Test) which takes one conjugacy class to the other.

   (b) $C_G(\sigma) = \langle(123), (14)(25)(36)\rangle$ of order 18. This subgroup cyclically permutes $1,2,3$ in any of three ways; also cyclically permuting $4,5,6$ in any of three ways; and also possibly switching the first three points 1,2,3 with the last three points 4,5,6. This gives a total of $3 \cdot 3 \cdot 2 = 18$ permutations. Written out explicitly, the elements of $C_G(\sigma)$ are

()	(123)	(132)	(14)(25)(36)	(142536)	(143625)
(456)	(123)(456)	(132)(456)	(152634)	(153426)	(15)(26)(34)
(465)	(123)(465)	(132)(465)	(163524)	(16)(24)(35)	(162435)

   where the first three columns list the elements of the elementary abelian 3-subgroup $\langle(123), (456)\rangle$ of order 9; and the rightmost three columns give its coset after multiplying by $(14)(25)(36)$.

   This subgroup is nonabelian; the two generators given above do not commute. Certainly it is not cyclic, since an elementary consideration of possible cycle structures in S_6 shows that this group has no elements of order 18.

   (c) Any of the three involutions highlighted in blue above have the required property.

2. Note that $\phi(g) = (g^T)^{-1} = (g^{-1})^T$. This is because transposing $gg^{-1} = g^{-1}g = I$ gives $(g^{-1})^T g^T = g^T (g^{-1})^T = I$. Mathematicians often simply denote $g \mapsto g^{-T}$ for this inverse-transpose map.

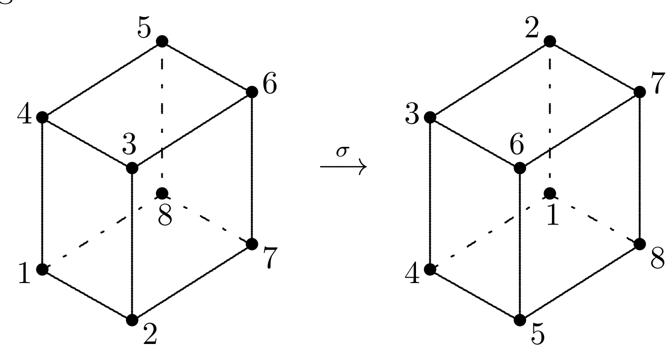
(a) Since $\phi(gh) = ((gh)^T)^{-1} = (h^T g^T)^{-1} = (g^T)^{-1}(h^T)^{-1} = \phi(g)\phi(h)$ for all $g, h \in G$. Moreover, ϕ is bijective since it has an inverse function, namely $\phi^{-1} = \phi$.

(b) No, in general ϕ is not inner. Every inner automorphism of G preserves determinant since $\det(wgw^{-1}) = \det g$ for all $g, w \in G$. However, $\det \phi(g) = (\det(g^T))^{-1} = (\det g)^{-1}$ for all $g \in G$. For $n \geqslant 1$, we may take $g \in G$ with determinant not equal to ± 1; then there is no inner automorphism taking g to $\phi(g)$.

(c) For $g \in SL_2(\mathbb{R})$, we have $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for some $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$. In order for w to conjugate g to $g^{-1} = \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$, we must have $wgw^{-1} = g^{-1}$, i.e. $wg = g^{-1}w$. This gives four linear equations for the four unknown entries of w; also we require $\det w = 1$. This system has a solution $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. (The solution is not unique: $-w$ is also a solution.)

Remark: For $n \geqslant 3$, the inverse-transpose map is an *outer* automorphism of $SL_n(\mathbb{R})$, an interesting fact with an elementary proof which however requires a little more thought to see.

3. (a) This shows that $\sigma = (143678)(25)$ is a symmetry of order $|\sigma| = 6$ of the cube satisfying the given conditions:



(b) By comparing $\sigma = (143678)(25)$ with $\sigma^{-1} = (187634)(25)$, we see that $\tau = (37)(48) \in G$ conjugates σ to σ^{-1}. Indeed $\tau \in G$, since it represents a reflection in the plane containing vertices 1,2,5,6. (This answer is not unique; multiplying τ by any power of σ gives other possible answers.)

(c) σ reverses orientation. For example, the edges 12, 14, 18 cycle counter-clockwise about the common vertex 1, when viewed from outside the original cube; but they cycle clockwise on the image cube.

(d) $C_G(\sigma) = \langle \sigma \rangle$ is cyclic of order 6. There are several ways to see this. Let $H = C_G(\sigma)$. Since $\langle \sigma \rangle \leqslant H$, it suffices to show that $|H| = 6$. Because of the action of $\sigma \in H$, the H-orbit of vertex 1 satisfies $\mathcal{O}_H(1) \supseteq \{1, 4, 3, 6, 7, 8\}$. But equality must hold, by our understanding of cycle structures of conjugates. (If $h \in H$, then h must conjugate σ to σ, so it must map the 6-cycle of σ to itself.) So $|\mathcal{O}_H(1)| = 6 = [H : H_1]$ where $H_1 = \mathrm{Stab}_H(1)$. If $h \in H_1$, then $1 = h(1) = h\sigma(8) = \sigma h(8)$ so $h(8) = \sigma^{-1}(1) = 8$. Similarly h fixes 7,6,3,4. This forces $h = ()$. So $|H| = [H : H_1]|H_1| = 6 \cdot 1 = 6$.

Alternatively, the conjugacy class of σ in G has size $[G : C_G(\sigma)] = \frac{48}{|C_G(\sigma)|}$, so it suffices to see that σ has exactly 8 conjugates in G. There are four pairs of antipodal vertices of the cube (such as $\{2, 5\}$). And for each such antipodal pair, there are two symmetries of order 6 interchanging the antipodal pair. (For example, every symmetry interchanging 2 and 5 is determined by how it maps the neighbors of 2 to the neighbors of 5. There are exactly six bijections $\{1, 3, 7\} \to \{4, 6, 8\}$. This gives σ, σ^{-1}, $(16)(25)(38)(47)$, $(14)(25)(38)(67)$, $(16)(25)(34)(87)$, $(18)(25)(36)(47)$ as all the elements of G interchanging $2 \leftrightarrow 5$.) This gives $4 \cdot 2 = 8$ conjugates of σ in G.

4. (a) The image of ϕ is the subgroup $\langle(132), (23)\rangle \cong S_3$. So no, ϕ is not surjective.

   (b) The map ϕ is 4-to-1 since $\frac{|S_4|}{|\phi(S_4)|} = \frac{24}{6} = 4$, so $|\ker \phi| = 4$. In fact, $\ker \phi = \langle(12)(34), (13)(24)\rangle$ is the normal Klein four-subgroup of S_4.

   (c) Yes, there is a unique homomorphism $\phi : S_4 \to S_4$ having the two values specified. This is simply because we defined ϕ on a pair of elements $(123), (34)$ which generate S_4.

5. See HW3.

   (a) $|G| = (5^2 - 1)(5^2 - 5) = 480$

   (b) $\left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ have order 5. These represent the fractional linear transformations $x \mapsto \frac{x}{x+1}$ and $x \mapsto x + 1$ respectively.

   (c) $\left(\begin{smallmatrix}0 & 1\\ -1 & -1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & 1\\ 4 & 4\end{smallmatrix}\right)$ has order 3; recall that this represents the fractional linear transformation $x \mapsto \frac{1}{1-x}$ of order 3. Alternatively, a linear transformation of order 3 is a root of $x^3 - 1 = (x - 1)(x^2 + x + 1)$ and not a root of $x - 1$; so we want g to have characteristic polynomial $x^2 + x + 1$. Such a matrix should have trace $-1 = 4$ and determinant 1. Our matrix (or others like it) can be found this way.

   A third solution is to let g permute three nonzero vectors cyclically; for example, take $g : \mathbf{v}_1 \mapsto \mathbf{v}_2 \mapsto \mathbf{v}_3 \mapsto \mathbf{v}_1$. This necessarily gives a linear transformation of order 3. To find the matrix of such a linear transformation with respect to the standard basis, for convenience we can take $\mathbf{v}_1 = \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$, $\mathbf{v}_2 = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$, $\mathbf{v}_3 = -\mathbf{v}_1 - \mathbf{v}_2 = \left[\begin{smallmatrix}4\\ 4\end{smallmatrix}\right]$. The matrix of such a linear transformation is $\left[\begin{smallmatrix}0 & 4\\ 1 & 4\end{smallmatrix}\right]$, which is again of the above form. Note that in order to complete a 3-cycle $g : \mathbf{v}_1 \mapsto \mathbf{v}_2 \mapsto \mathbf{v}_3 \mapsto \mathbf{v}_1$ where $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$, we must have $\mathbf{v}_1 = g\mathbf{v}_3 = ag\mathbf{v}_1 + bg\mathbf{v}_2 = a\mathbf{v}_2 + b(a\mathbf{v}_1 + b\mathbf{v}_2)$; so we require $ab = 1$ and $a + b^2 = 0$. We solve to obtain $a = b = -1 = 4$.

   (d) If g_3 has order 3, then $g = -g_3$ has order 6. For example from (c), we get $\left(\begin{smallmatrix}0 & 4\\ 1 & 1\end{smallmatrix}\right)$ of order 6. Alternatively, we want g to be a root of $x^6 - 1 = (x + 1)(x - 1)(x^2 + x + 1)(x^2 - x + 1)$. Ruling out elements of order 1, 2 and 3, we want g to have characteristic polynomial $x^2 - x + 1$. For this, we want g to have trace 1 and determinant 1. Our matrix is one of many with this form.

6. (a) T (b) F (c) F (d) T (e) T (f) F (g) F (h) T (i) T (j) T

You are not required to provide explanations; but I do so here for your benefit:

(a) As discussed in class, this follows from Cayley's Representation Theorem, since S_n is isomorphic to a subgroup of $GL_n(\mathbb{R})$ (the subgroup of all $n \times n$ permutation matrices).

(b) If C_2 is cyclic of order 2, then $C_2 \times C_2 \times C_2$ (the elementary abelian group of order 8) cannot be generated by any two of its elements.

(c) Counterexample: The direct product $S_3 \times S_3$ of order 36 has a subgroup H of order 18 consisting of all pairs $(\sigma, \tau) \in S_3 \times S_3$ such that σ and τ are either both even, or both odd. This is not equal to $A \times B$ for any subgroups $A, B \leqslant S_3$. (In fact, since H has no elements of order 6, it is not even *isomorphic* to any subgroup of the form $A \times B$.)

(d) As discussed in class, a group is dihedral iff it is generated by two involutions.

(e) This is easy to prove (using the hypothesis that G is abelian; without that hypothesis, the statement fails).

(f) This is another fact we have discussed before. Consider the set of all elements $z \in \mathbb{C}^\times$ of finite order, i.e. all complex roots of unity; this is an infinite abelian group in which all elements have finite order.

(g) Recall that A_5 is simple (its only normal subgroups are the trivial subgroup $\{()\}$ and A_5 itself).

(h) Every group of prime order is cyclic. Recall that this is an elementary consequence of Lagrange's Theorem. If G is a group of prime order p, then let g be any nonidentity element of G; since $\langle g \rangle$ is a nontrivial subgroup of order dividing p, we have $|\langle g \rangle| = p$, so $G = \langle g \rangle$ is cyclic.

(i) The symmetry group of a regular n-gon centered at the origin has a cyclic subgroup generated by a rotation of angle $\frac{2\pi}{n}$. This rotation is a linear transformation in $SL_2(\mathbb{R})$.

(j) The subgroup H can be uniquely recovered from any of its left cosets by $\{x^{-1}y : x, y \in gH\} = H$, as is easy to verify.