Algebra I

Group Theory

Book 1

A *group* is a set G with a binary operation $*$ which has an identity element; the operation is associative; and every element has an inverse.

Eg. $\mathbb{R}$ = set of real numbers under addition '+'. Its identity element is 0.

$$\left.\begin{array}{l} 0 + x = x \\ (x+y)+z = x+(y+z) \\ x+(-x) = 0 = (-x)+x \end{array}\right\} \quad \text{for all } x,y,z \in \mathbb{R}$$

$(\mathbb{R}, +)$ is a group.

$(\mathbb{R}, \times)$ (real numbers under multiplication) is almost but not quite a group. (0 does not have an inverse). 1 is the identity.

$\mathbb{R}^\times$ = {all nonzero real numbers} $= \{a \in \mathbb{R} : a \neq 0\}$ is a group under multiplication.

$$\left.\begin{array}{l} 1a = a \\ (ab)c = a(bc) \\ a \cdot a^{-1} = a^{-1} \cdot a = 1 \qquad a^{-1} = \frac{1}{a} \end{array}\right. \quad \text{for all } a,b,c \in \mathbb{R}^\times.$$

$(\mathbb{R}^\times, \times)$ is a group.

$\mathbb{R}$ with the operation $x * y = x + y + 7$. This is a group $(\mathbb{R}, *)$. For all $x,y,z \in \mathbb{R}$,

$$(x*y)*z = (x+y+7)+z+7 = x+y+z+14 = x+(y+z+7)+7 = x*(y*z)$$

so $(\mathbb{R}, *)$ is associative. Note that $-7 \in \mathbb{R}$ is an identity element since

$$\left.\begin{array}{l} -7 * x = (-7)+x+7 = x \\ \text{and} \quad x*(-7) = x+(-7)+7 = x \end{array}\right\} \quad \text{for all } x \in \mathbb{R}.$$

So $-7 \in \mathbb{R}$ is an identity element for '$*$'.

$$\left.\begin{array}{l} (-x-14)*x = (-x-14)+x+7 = -7 \\ x*(-x-14) = x+(-x-14)+7 = -7 \end{array}\right\} \quad \text{for all } x \in \mathbb{R}.$$

So $-x-14$ is an inverse element for x.

$$(x*y)*z = x*(y*z)$$
$$\Leftrightarrow (x+y+7)+z+7 = x+(y+z+7)+7$$
$$\Leftrightarrow x+y+z+14 = x+y+z+14$$

so $(\mathbb{R}, *)$ is associative.

$$7 = 3$$
$$\Rightarrow 7-5 = 3-5$$
$$\Rightarrow 2 = -2$$
$$\Rightarrow (2)^2 = (-2)^2$$
$$\Rightarrow 4 = 4$$

$$(x*y)*z = (x+y+7)+z+7$$
$$= x+y+z+14$$
$$= x+(y+z+7)+7$$
$$= x*(y*z)$$

$(\mathbb{Q}, +)$ is a group. $\mathbb{Q} = \{\text{rational numbers}\}$

$-\frac{5}{3} \in \mathbb{Q}$

$\frac{172}{100} = 1.72 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

$(\mathbb{Q}^\times, \times)$ is a group.

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\} = \{\text{all nonzero rational numbers}\}$

$(\mathbb{N}, +)$ is not a group

$\mathbb{N} = \{1,2,3,4,\cdots\} = \mathbb{Z}^{>0}$

$\mathbb{N}_0 = \{0,1,2,3,4,\cdots\} = \mathbb{Z}^{\geq 0}$

$\mathbb{Z} = \{\text{integers}\} = \{\cdots, -3,-2,-1,0,1,2,3,4,\cdots\}$

$(\mathbb{Z}, +)$ is a group.