Algebra I

Group Theory

Book 1

A <u>group</u> is a set G with a binary operation $*$ which has an identity element; the operation is associative; and every element has an inverse.

Eg. $\mathbb{R}$ = set of real numbers under addition '+'. Its identity element is 0.

$$\left.\begin{aligned} &0 + x = x \\ &(x+y)+z = x+(y+z) \\ &x+(-x) = 0 = (-x)+x \end{aligned}\right\} \quad \text{for all } x,y,z \in \mathbb{R}$$

$(\mathbb{R}, +)$ is a group.

$(\mathbb{R}, \times)$ (real numbers under multiplication) is almost but not quite a group. (0 does not have an inverse). 1 is the identity.

$\mathbb{R}^\times$ = {all nonzero real numbers} $= \{a \in \mathbb{R} : a \neq 0\}$ is a group under multiplication.

$$\begin{aligned} &1a = a \\ &(ab)c = a(bc) \\ &a \cdot a^{-1} = a^{-1} \cdot a = 1 \qquad a^{-1} = \tfrac{1}{a} \end{aligned} \qquad \text{for all } a,b,c \in \mathbb{R}^\times.$$

$(\mathbb{R}^\times, \times)$ is a group.

$\mathbb{R}$ with the operation $x * y = x + y + 7$. This is a group $(\mathbb{R}, *)$. For all $x,y,z \in \mathbb{R}$,

$$(x*y)*z = (x+y+7)+z+7 = x+y+z+14 = x+(y+z+7)+7 = x*(y*z)$$

so $(\mathbb{R}, *)$ is associative. Note that $-7 \in \mathbb{R}$ is an identity element since

$$\left.\begin{aligned} &-7 * x = (-7)+x+7 = x \\ \text{and} \quad &x*(-7) = x+(-7)+7 = x \end{aligned}\right\} \quad \text{for all } x \in \mathbb{R}.$$

So $-7 \in \mathbb{R}$ is an identity element for '$*$'.

$$\left.\begin{aligned} &(-x-14)*x = (-x-14)+x+7 = -7 \\ &x*(-x-14) = x+(-x-14)+7 = -7 \end{aligned}\right\} \quad \text{for all } x \in \mathbb{R}.$$

So $-x-14$ is an inverse element for x.

$$(x*y)*z = x*(y*z)$$
$$\Leftrightarrow (x+y+7)+z+7 = x+(y+z+7)+7$$
$$\Leftrightarrow x+y+z+14 = x+y+z+14$$

so $(\mathbb{R}, *)$ is associative.

$$7 = 3$$
$$\Rightarrow 7-5 = 3-5$$
$$\Rightarrow 2 = -2$$
$$\Rightarrow (2)^2 = (-2)^2$$
$$\Rightarrow 4 = 4$$

$$\begin{aligned}(x*y)*z &= (x+y+7)+z+7\\ &= x+y+z+14\\ &= x+(y+z+7)+7\\ &= x*(y*z)\end{aligned}$$

$(\mathbb{Q}, +)$ is a group. $\mathbb{Q} = \{\text{rational numbers}\}$

$-\frac{5}{3} \in \mathbb{Q}$

$\frac{172}{100} = 1.72 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

$(\mathbb{Q}^\times, \times)$ is a group.

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\} = \{\text{all nonzero rational numbers}\}$

$(\mathbb{N}, +)$ is not a group

$\mathbb{N} = \{1, 2, 3, 4, \cdots\} = \mathbb{Z}^{>0}$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \cdots\} = \mathbb{Z}^{\geq 0}$

$\mathbb{Z} = \{\text{integers}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$(\mathbb{Z}, +)$ is a group.

$(\mathbb{Z}, +) \leq (\mathbb{Q}, +) \leq (\mathbb{R}, +) \leq (\mathbb{C}, +)$; but $(\mathbb{R}^\times, \times)$ is ~~not~~ a subgroup $(\mathbb{R}, +)$ (although $\mathbb{R}^\times \subseteq \mathbb{R}$)

$\not\leq$

subgroup subgroup subset

In $\mathbb{R}^\times$, $2 \times 3 = 6$ but in $(\mathbb{R}, +)$, $2 + 3 = 5$

$GL_n(\mathbb{R}) = \{$ invertible $n\times n$ matrices with real entries $\}$ is the <u>general</u> <u>linear</u> <u>group</u>

$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$GL_n(\mathbb{R})$ is a multiplicative group with identity $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$GL_n(\mathbb{R})$ is <u>not</u> commutative for $n \geq 2$.

$GL_1(\mathbb{R})$ is commutative.

$(G, *)$ is <u>Abelian</u> (abelian) if $x * y = y * x$ for all $x, y \in G$.

$GL_n(\mathbb{R})$ is abelian for $n=1$; nonabelian for $n \geq 2$. $\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 5 & 35 \end{bmatrix}$ whereas $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 38 \end{bmatrix}$.

$GL_1(\mathbb{R}) \cong \mathbb{R}^\times$ (these are isomorphic groups i.e. essentially the same group. Since $\mathbb{R}^\times$ is abelian, so is $GL_1(\mathbb{R})$.)

Function composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "g \circ h"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & Z \arrow[r, "f"] & W \end{tikzcd}$$

If $x \in X$ then $h(x) \in Y$, $g(h(x)) \in Z$, $\underbrace{f(g(h(x)))}_{(f\circ g\circ h)(x)} \in W$.

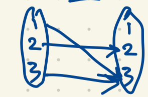
$f \circ g \neq g \circ f$

Because matrix multiplication is expressing the composition of linear transformations, it is associative but not necessarily commutative.

If X is any set, the bijections $X \xrightarrow{f} X$ (i.e. f one-to-one and onto) form a group under composition. This is the <u>symmetric group</u>

$$G = \mathrm{Sym}\,X = \{\text{bijections } X \to X\} = \{\text{permutations of } X\}.$$

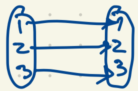
Not a bijection (neither one-to-one nor onto)



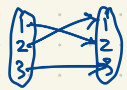
eg. $X = [3] = \{1,2,3\}$. (Notation: $[n] = \{1,2,3,\dots,n\}$.)

There are exactly $3! = 6$ bijections $[3] \to [3]$.

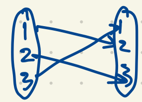
$n! = 1 \times 2 \times 3 \times \cdots \times n$ (n factorial) is the number of permutations of $[n]$.



x	$f(x)$
1	1
2	2
3	3



x	$f(x)$
1	2
2	1
3	3



$|S_3| = 6$. S_3 is a nonabelian group of order 6.

S_3 is the smallest nonabelian group.

In S_3, $(12)(13) = (132)$
$(13)(12) = (123)$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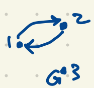
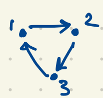
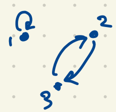
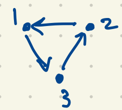
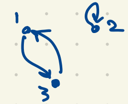

(1 2)


(1 2 3)


(2 3)

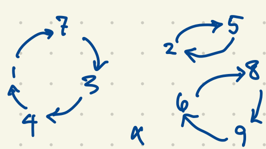

(1 3 2)


(1 3)

cycle notation for $\mathrm{Sym}[3] = S_3$

$= \{(\,), (12), (13), (23), (123), (132)\}$ (12)(3)

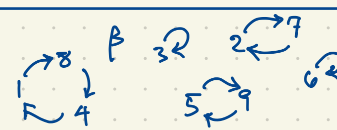
Eg. $n = 9$



α

n	$\alpha(n)$	$\beta(n)$	$\alpha\beta(n)$
1	7	8	9
2	5	7	3
3	4	3	4
4	1	1	7
5	2	9	6
6	8	6	8
7	3	2	5
8	9	4	1
9	6	5	2

β



$\beta = \underbrace{(7,2)}\underbrace{(4,1,8)}(3)(6)(5,9) = (1\ 8\ 4)(2\ 7)(5\ 9)$

$(7,2) = (2,7)$

$(4,1,8) = (1,8,4) = (8,4,1)$

$(3) = (\,)$

$\alpha = (1,7,3,4)(2,5)(6,8,9)$

$\alpha\beta = \alpha\circ\beta = (1,9,2,3,4,7,5,6,8) = (1734)(25)(689)(184)(27)(59)$

$\beta\alpha = \beta\circ\alpha = (129648573) = (184)(27)(59)(1734)(25)(689)$

If α, β are permutations then $\alpha\beta \neq \beta\alpha$ in general but they have the same cycle structure.

The order of a group G is $|G|$, the number of elements in the group. (finite or infinite)

$|S_n| = n!$

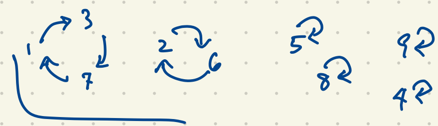
$|GL_n(\mathbb{R})| = \infty$

S_n is nonabelian for $n \geq 3$.

$S_2 = \{(), (12)\}$ is abelian.

In S_n, disjoint cycles always commute, e.g. in S_9, $(137)(26) = (26)(137)$

If two permutations commute, must they have disjoint cycles?



$\alpha = (135)(246)$

$\beta = (12)(34)(56)$

Note: The two 3-cycles in α intersect with the three 2-cycles in β.

$\alpha\beta = (135)(246)(12)(34)(56) = (145236)$

$\beta\alpha = (12)(34)(56)(135)(246) = (145236)$

S_n acts on $[n] = \{1, 2, \dots, n\}$ (the n points that we are permuting)

Do not confuse S_n with $[n]$. THIS IS NOT THAT. $|S_n| = n!$, $|[n]| = n$.

Typically, groups act on things (generically called points).

Typically, groups describe symmetries of things.

A cube has 48 symmetries forming a group G of order 48. $|G| = 48$.

24 of these are direct symmetries preserving orientation: these are rotations.

24 of these are virtual symmetries which cannot be obtained by physical motion.

number of edges

$12 \times 4 = 48$

number of symmetries fixing each edge

number of vertices

$8 \times 6 = 48$

number of symmetries fixing each vertex

how many symmetries map each face to each of the other faces

$6 \times 8 = 48$

number of faces

In a group G with identity e, an element $g \in G$ has <u>order</u> n if $\underbrace{g^n}_{g * g * \cdots * g,\ n \geq 1} = e$
but no smaller power of g equals e.

If G is the symmetry group of a cube, every reflection has order 2.

Also a $180°$ rotation about any axis has order 2.

A $120°$ rotation of the cube about an axis joining two opposite (antipodal) vertices has order 3.

The cube has axes of symmetry joining centers of opposite faces, and a $90°$ rotation around such an axis has order 4.

In any group, the identity has order 1.

S_3 has
- 1 element of order 1, i.e. $()$
- 3 elements of order 2, i.e. (12), (13), (23)
- 2 elements of order 3, i.e. (132), (123)

$|S_3| = 6$

The order of an n-cycle. If $\alpha = (1,2,3,\dots,n)$ then $\alpha^n = ()$ but $\alpha^k \neq ()$ for $k = 1,2,\dots,n-1$.

S_4 has
- 1 element of order 1, i.e. $()$
- 9 elements of order 2, i.e. $(12), \dots, (13)(24), \dots$ (six $= \binom{4}{2}$ 2-cycles $(i\,j)$; three permutations $(i\,j)(k\,l)$ having the same cycle structure as $(13)(24)$)
- 8 elements of order 3, i.e. $(123), \dots$ (eight 3-cycles $(i\,j\,k)$, the same cycle structure as (123))
- 6 elements of order 4, six 4-cycles e.g. (1234)

$|S_4| = 24$

$$\binom{m}{n} = \binom{m}{m-n}$$

$\binom{m}{n}$ = number of n-subsets of an m-set, e.g. $\binom{4}{2} = 6$: a 4-set (set with 4 elements, e.g. $[4] = \{1,2,3,4\}$) has six subsets of size 2 : $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

$$= \frac{m!}{n!\,(m-n)!} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n(n-1)(n-2)\cdots 1}$$

so $\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$.

$\binom{4}{3} \cdot 2 = 4 \cdot 2 = 8$

$\binom{4}{3} = \frac{4 \cdot \cancel{3} \cdot \cancel{2}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 4$

$S_5 = \{\text{permutations of } [5] = \{1,2,3,4,5\}\}$ is a group of order $|S_5| = 5! = 120$.

$(1\ 2)(1\ 3) = (1\ 3\ 2)$

How many elements of each order does S_5 have?

$\underline{1}$ element of order 1: $(\,)$

$\underline{25}$ elements of order 2: $(i\ j)$ $\binom{5}{2} = 10$ cycles of length 2

$(i\ j)(k\ l)$ $5 \times 3 = 15$ elements which are a product of two disjoint 2-cycles

or: $\underbrace{10}_{\text{how many choices of 2-cycle } (i\ j)} \times \underbrace{3}_{\text{how many 2-cycles } (k\ l) \text{ disjoint from } (i\ j)} \div 2 = 15$

since $(i\ j)(k\ l) = (k\ l)(i\ j)$

A 2-cycle $(i\ j)$ (i.e. cycle of length 2) is a <u>transposition</u>.

$\underline{20}$ elements of order 3: 3-cycles $(i\ j\ k)$ $\binom{5}{3} \times 2 = 10 \times 2 = 20$

$\underline{30}$ elements of order 4: 4-cycles $(i\ j\ k\ l)$ e.g. $(1\ 2\ 3\ 4)$, $(1\ 3\ 4\ 2)$, $(2\ 5\ 3\ 4)$, ...

$\underline{24}$ elements of order 5: 5-cycles $(1 \underbrace{* * * *}_{2,3,4,5})$

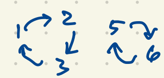
$\binom{5}{4} \times 3! = 5 \times 6 = 30$

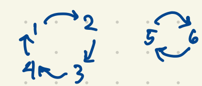
how many ways to choose i, j, k, l

$\underline{20}$ elements of order 6: $(i\ j\ k)(l\ m)$

$(1\ 2\ 3)(4\ 5) \in S_5$ has order 6

$(1\ 2\ 3\ 4)(5\ 6) \in S_6$ has order 4





$120 = |S_5|$

If $\alpha \in S_n$ is written as a product of disjoint cycles, then its order is the least common multiple of the lengths of its cycles.

$(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$ has order 15

$(1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9)$ ⋯⋯ 6

In $\mathbb{R}^\times = \{\text{nonzero real numbers}\}$ under multiplication,

1 has order 1;

-1 " " 2; $(-1 \cdot -1 = 1)$

every other element of $\mathbb{R}^\times$ has infinite order.

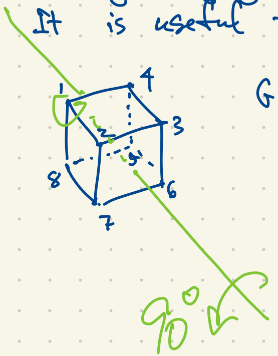
If $a \in \mathbb{R}^\times$, $\operatorname{ord}(a) = \begin{cases} 1, & \text{if } a = 1; \\ 2, & \text{if } a = -1; \\ \infty, & \text{otherwise.} \end{cases}$

We also write the order of $a \in G$ as $|a|$ e.g.

$|(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)| = 15$

$\operatorname{ord}((1\ 2\ 3)(4\ 5\ 6\ 7\ 8)) = 15$

The symmetry group of a cube is a group G of order 48 ie. $|G| = 48$.
It is useful to think of G as a subgroup of S_8:



$$G = \{ (\,), \underbrace{(1234)(5876)}_{\substack{\text{90° rotation about} \\ \text{the vertical axis} \\ \text{of symmetry}}}, \underbrace{(1854)(2763)}_{\substack{\text{90° rotation} \\ \text{about green} \\ \text{axis of symmetry}}}, \underbrace{(18)(27)(36)(45)}_{\substack{\text{reflection in} \\ \text{horizontal plane} \\ \text{of symmetry}}}, (173)(486), \dots \}$$

identity

$(1854)(2763)(1234)(5876) = (173)(2)(486)(5) = (173)(486)$ is a $120°$ rotation about the axis joining the pair of antipodal vertices 2, 5

If G is any group and $g_1, \dots, g_k \in G$ then $\boxed{\langle g_1, g_2, \dots, g_k \rangle = \text{the subgroup of } G \text{ generated by } g_1, \dots, g_k}$ ie. the smallest subgroup of G containing $g_1, \dots, g_k$.

The letter S has a rotational symmetry about its centre (rotate 180° about → S. The symmetry group in this case is $\{I, R\}$ where R is the 180° rotation, $R^2 = I$. Both symmetries of S preserve orientation.

S ≠ S

U ≠ N

U has symmetry group of order 2 $\{I, T\}$ where T is a reflection in the vertical axis of symmetry, $T^2 = I$

Reflections reverse orientation; rotations preserve orientation.

Y has symmetry group of order 2

Y has symmetry group of order 1.

Y has symmetry group of order 6. (3 rotational symmetries, 3 reflective symmetries).

For any object $X \subset \mathbb{R}^n$, either all symmetries of X preserve orientation or exactly half of the symmetries preserve orientation (so the other half reverse orientation).

reflection about vertical axis of symmetry

" "

axis $\cancel{Y}$

The symmetry group of Y is $\{\underbrace{I, R}, R^2, T, TR, TR^2\} = \langle T, R \rangle$

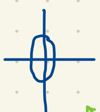
(counter-clockwise 120° rotation about center)

$\cancel{Y}$

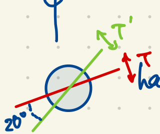
$RT = TR^2$ so the group is nonabelian.

$E \neq E$

The figure E∃ as a symmetry group of order 4 $\{I, R, T, RT\}$ where I = identity, R = 180° rotation about the center, T = reflection in horizontal axis of symmetry, $RT = TR$ = reflection in the vertical axis of symmetry. This group is abelian.



O has the same symmetry group as E∃ (abelian of order 4).



has infinitely many symmetries. The symmetry group is infinite nonabelian.

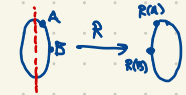
$\underbrace{TT'}_{\text{40° rotation clockwise}} \neq \underbrace{T'T}_{\text{40° rotation counterclockwise about center}}$

40° rotation clockwise (i.e. 320° counterclockwise rotation about center)

40° rotation counterclockwise about center

A symmetry of X is a bijection $X \to X$ (permutation of the points of X) which preserves distances and angles i.e. the shape of X. Here typically $X \subseteq \mathbb{R}^2$ or $X \subseteq \mathbb{R}^3$.
eg. if $X = \mathbb{R}^2$ then the symmetries (isometries) of $X = \mathbb{R}^2$ includes infinitely many transformations (rotations, reflections, translations, etc.).

If X is a circle then X has infinitely many symmetries.
If X is the pattern E then X has exactly 2 symmetries.
" " " " " " O then X " " 4 " .



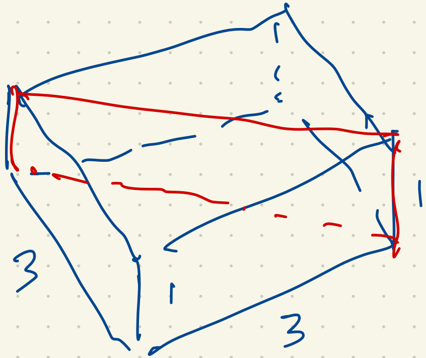
The letter R has trivial symmetry group (only the identity).
" " S has symmetry group of order 2.
Note: The symmetry group of X is a subgroup of $\mathrm{Sym}\,X = \{$all permutations of $X\}$.
Eg. the pattern $\ldots SSSSS \ldots$ in $\mathbb{R}^2$ is different from its mirror images so all its symmetries are orientation-preserving (in particular it has no reflective symmetries).
Some symmetries of the pattern:

For every point at the center of some S, rotate 180° about that point.
Also we have translational symmetries found by translating an integer distance horizontally.
Also, half turns about any point midway between the centers of two adjacent S's.

$RR' \neq R'R$

$RR'(A) \neq R'R(A)$ so $RR' \neq R'R$ so the symmetry group is nonabelian.

In fact RR' is a translation two units to the left where a 'unit' is the distance between the centers of two adjacent S's. And $R'R$ is the translation two units to the right. $(R'R)^{-1} = RR'$

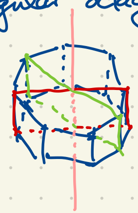


$1 \times 3 \times 3$ block

has 16 symmetries.

Compare: A square has only 8 symmetries.

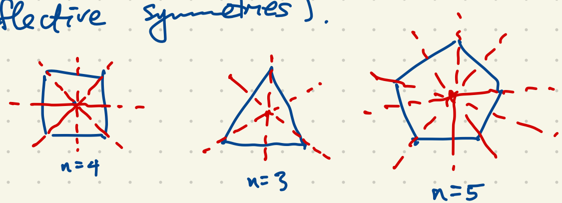
A regular octagonal prism has symmetry group of order 32. This group is nonabelian.



(16 rotational symmetries and 16 other symmetries which reverse orientation).

A regular n-gon ($n \geq 3$) has a symmetry group of order $2n$ (n rotational symmetries and n reflective symmetries).

Dihedral groups



In $\mathbb{R}^\times$, the multiplicative group of nonzero real numbers,

$$\langle 3\rangle = \{\ldots, \tfrac{1}{27}, \tfrac{1}{9}, \tfrac{1}{3}, 1, 3, 9, 27, 81, 243, \cdots\} = \{3^k : k\in\mathbb{Z}\}$$

$\langle 2,3\rangle = \{2^k 3^l : k,l\in\mathbb{Z}\}$ so $\frac{2}{9}\in\langle 2,3\rangle$, $5\notin\langle 2,3\rangle$, $25\notin\langle 2,3\rangle$ (non-cyclic but it is abelian)

$\langle -1\rangle = \{1,-1\}$

$\langle 1\rangle = \{1\}$

<u>Theorem</u> Let G be a group with identity 1 and let $g\in G$. Then $\langle g\rangle = \{g^k : k\in\mathbb{Z}\}$ has order $|\langle g\rangle| = |g|$.
(The order of each element is the order of the subgroup that it generates.)

A subgroup that is generated by a single element (i.e. a subgroup of the form $\langle g\rangle$ for some $g\in G$) is called <u>cyclic</u>. Cyclic groups (i.e. groups that are generated by a single element) are always abelian since in $\langle g\rangle = \{g^k : k\in\mathbb{Z}\}$ we have $g^i g^j = g^{i+j} = g^j g^i$.

In the subgroup $\langle 3\rangle < \mathbb{R}^\times$ has two generators $3, \frac{1}{3}$ since $\langle 3\rangle = \langle \frac{1}{3}\rangle$.

$\langle 3\rangle \subset \mathbb{R}^\times$

$\mathbb{R}^\times$ is not finitely generated: there is no finite list of elements $a_1,\cdots,a_k\in\mathbb{R}^\times$ such that $\langle a_1,\cdots,a_k\rangle = \mathbb{R}^\times$. For every finite list $a_1,\cdots,a_k\in\mathbb{R}^\times$, the subgroup $\langle a_1,\cdots,a_k\rangle < \mathbb{R}^\times$ is a <u>proper subgroup</u> (i.e. a subgroup which is a proper subset). $H\leq G$ means H is a subgroup of G; $H<G$ means H is a proper subgroup of G.

<u>Proof of the Theorem</u> (about orders)

First suppose $g\in G$ has infinite order i.e. $g^k\neq 1$ for $k=1,2,3,\cdots$. We must show that $|\langle g\rangle| = \infty$ where $\langle g\rangle = \{g^k : k\in\mathbb{Z}\}$. We will prove that all the powers g^k $(k\in\mathbb{Z})$ are distinct in this case. If not, then $g^k = g^l$ for some $k,l\in\mathbb{Z}$ with $k\neq l$, then without loss of generality $k<l$ and $1 = g^0 = g^{-k}g^k = g^{-k}g^l = g^{l-k}$, a contradiction.
So $|\langle g\rangle| = \infty$ in this case.

Next suppose $|g| = n$ is finite i.e. n is a positive integer and $g^k \neq 1$ for $k = 1, 2, 3, \dots, n-1$ but $g^n = 1$. We will show that $\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ where these n elements are distinct. The same argument as above shows that $1, g, g^2, \dots, g^{n-1}$ are distinct (otherwise $g^k = g^l$ with $1 \le k < l \le n-1$ and then $g^{l-k} = 1$ where $l-k \in \{1, 2, \dots, n-1\}$, contrary to the assumption $|g| = n$. It remains to show that $g^k \in \{1, g, g^2, \dots, g^{n-1}\}$ for every $k \in \mathbb{Z}$. For this we use the Division Algorithm: $k = qn + r$ where $q, r \in \mathbb{Z}$, $r \in \{0, 1, 2, \dots, n-1\}$.

Then $g^k = g^{qn+r} = (g^n)^q \cdot g^r = 1^q \cdot g^r = g^r \in \{1, g, g^2, \dots, g^{n-1}\}$. $\square$

Algebra 9/27/23

In $S_3 = \{(1), (12), (13), (14), (123), (132)\}$

the subset $\{(1), (12), (13)\}$ is not a subgroup

It is not a group since multiplication is not function a binary operation. A binary operation on S is a map $S \times S \to S$ where $S \times S$ is $= \{(s,t) \mid s, t \in S\}$

For an operation $* : S_3 \times S_3 \to S_3$, $*(g,h) \in S_3$ is usually writen as $g * h$

In some books, we emphasize the property $g * h \in S$ by saying "$*$" is closed on S.

$\longrightarrow$ $f\ \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$

$\longmapsto$ $(g,h) \mapsto gh$

Examples of subgroups

$\{()\} < S_3$ $\langle () \rangle$ "all powers of the element"

the whole group

$\langle (12) \rangle =$

$\langle (123) \rangle = \{(), (123), (132)\}$

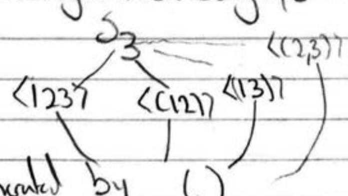
$|S_3| = 6$

(All its elements have order dividing 6 (elements have 1,2,3)

Lagrange's theorem says every subgroup $H \leq G$
(where G is a finite group) order of $H \div$ order G
$(|H| \mid |G|)$

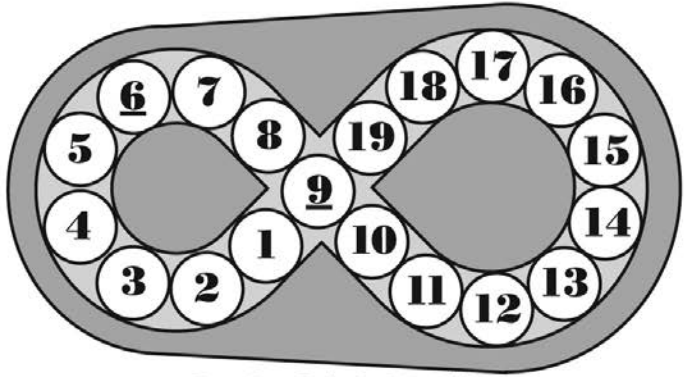
In particular, for every $g \in G$, $|g| \mid |G|$

Hasse Diagram of subgroups of S_3

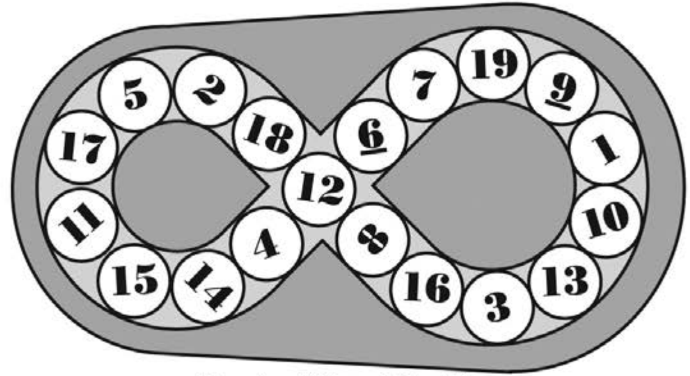


S_n has $\binom{n}{2}$ transpositions
(2 cycle (ij))
These generate S_n ie
$\langle (12), (13), \ldots, (1n) \rangle = S_n$

So S_n is generated by the $n-1$ transposition $(1j)$, $2 \leq j \leq n$



Puzzle: Original Position



Puzzle: Altered Position

The group of this puzzle is a subgroup of S_{19}.

Note: $|S_{19}| = 19! = 121645100408832000$

$\alpha = (1,2,3,\dots,9)$, $\beta = (9,10,11,\dots,19)$

$G = \langle \alpha, \beta \rangle$ = products of powers of α and β

$G < S_{19}$, $|G| = \frac{1}{2}|S_{19}| = 60822550204416000$

G consists of all <u>even</u> permutations of $1,2,3,\dots,19$.

For $n \geq 2$, exactly half of the permutations of $1,2,\dots,n$ are even and the other half are odd.

$G = A_{19}$

α is even
β is even

even · even = even
even · odd = odd
odd · odd = even

(just like addition of integers)

Classic 15 puzzle

which permutations are even? and which are odd?

Fix $n \geqslant 2$ and consider the polynomial $\Delta(x_1, \cdots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$

For $n=3$: $\Delta(x_1, x_2, x_3) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

For every $\alpha \in S_3$, $\alpha\Delta = \pm \Delta$ e.g. $(1\ 2)\Delta = (x_1 - x_2)(x_3 - x_2)(x_3 - x_1) = -(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

$= -\Delta$

$(1\ 3\ 2)\Delta = (1\ 3\ 2)(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = (x_1 - x_3)(x_2 - x_3)(x_2 - x_1) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$

$= \Delta$.

In general for $n \geqslant 2$, $\alpha \in S_n$, $\alpha\Delta(x_1, \cdots, x_n) = \underbrace{(\operatorname{sgn} \alpha)}_{\pm 1} \Delta$

If $\alpha, \beta \in S_n$ then $\alpha\Delta = (\operatorname{sgn} \alpha)\Delta$

$\beta\Delta = (\operatorname{sgn} \beta)\Delta$

$(\alpha\beta)\Delta = \operatorname{sgn}(\alpha\beta)\Delta = \alpha(\beta\Delta) = \alpha(\underbrace{(\operatorname{sgn} \beta)}_{\pm 1}\Delta) = (\operatorname{sgn} \beta)\alpha\Delta = (\operatorname{sgn} \beta)(\operatorname{sgn} \alpha)\Delta$

$(\operatorname{sgn} \alpha)(\operatorname{sgn} \beta) = \operatorname{sgn}(\alpha\beta)$.

α is $\begin{cases} \text{even} \\ \text{odd} \end{cases}$ if $\operatorname{sgn} \alpha = \begin{cases} +1 \\ -1 \end{cases}$

In S_3, $(1\,2), (1\,3), (2\,3)$ are odd

$(\ ), (1\,2\,3), (1\,3\,2)$ are even

In S_n, $n \geqslant 2$, the subset of all even permutations is a subgroup

$A_n < S_n$, $|A_n| = \frac{n!}{2}$ called the alternating group of degree n

S_n is the symmetric group of degree n.

Transpositions $(i\,j)$ are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd; and vice versa.
A cycle of odd length is an even permutation;
〃 〃 〃 even 〃 〃 〃 odd 〃 .

If α is a product of an even numbers of transpositions, then α is an even permutation.
〃 〃 〃 〃 〃 〃 〃 odd 〃 〃 〃 〃 〃 〃 〃 〃 odd 〃 .