Algebra I

Group Theory

Book 3

A matrix in $GL_2(\mathbb{R})$ is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ iff it has trace 0 and determinant -1.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$ then A has characteristic polynomial $f(x) = \det(xI - A) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$$= \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = (x-a)(x-d) - bc = x^2 - \underbrace{(a+d)}_{\operatorname{tr} A} x + \underbrace{(ad-bc)}_{\det A}.$$

Some books define the characteristic polynomial of A as $\det(A - xI) = (-1)^n \underbrace{\det(xI - A)}_{\substack{\text{monic:} \\ \text{its leading} \\ \text{term is } x^n}}$

Cayley-Hamilton Theorem (look it up in any linear algebra book)

If $f(x)$ is the characteristic polynomial of an $n \times n$ matrix A, then $f(A) = 0$.

In the 2×2 case, $A^2 - (\operatorname{tr} A) A + (\det A) I = 0$ holds as we compute here:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$A^2 - (\operatorname{tr} A) A + (\det A) I = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cancel{a^2}+\cancel{bc}-(\cancel{a}+\cancel{d})a+(\cancel{ad}-\cancel{bc}) & \cancel{ab}+\cancel{bd}-(\cancel{a}+\cancel{d})b \\ ac+cd-(a+d)c & bc+d^2-(a+d)d+(ad-bc) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If $A \in GL_2(\mathbb{R})$ has trace 0 and determinant -1 then it satisfies $A^2 - \underline{0}A - 1I = 0$ so $A^2 = I$

so in the group $GL_2(\mathbb{R})$, A has order ~~1 or~~ 2. ($\operatorname{tr} I = 2$, not 0)

$f(x) = \det(xI - A)$ may or may not be the smallest degree polynomial that has A as a root.

The <u>minimal polynomial</u> of A, $m(x)$, is the monic polynomial of smallest degree satisfying $m(A) = 0$.

Facts (see a linear algebra book):

- Roots of $f(x)$ are eigenvalues of A.
- $m(x)$ divides $f(x)$ i.e. $f(x) = h(x)\, m(x)$ for some monic polynomial $h(x)$ (often $h(x) = 1$, $m(x) = f(x)$).
- Every eigenvalue of A is a root of $m(x)$.

<u>Theorem</u> Let $A \in GL_2(\mathbb{R})$. Then the following are equivalent:

(i) $\operatorname{tr} A = 0$, $\det A = -1$

(ii) A has order 2 but $A \neq -I$.

(iii) A is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We have proved (i) $\Rightarrow$ (ii). And (iii) $\Rightarrow$ (i) is easy. Assume $A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ for some $M \in GL_2(\mathbb{R})$.

Then $\operatorname{tr} A = \operatorname{tr}\left(M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}\right) = \operatorname{tr}\left(M^{-1}M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \operatorname{tr}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0$.

$\operatorname{tr} AB = \operatorname{tr} BA$ if A is $m \times n$, B is $n \times m$ (short proof: see linear algebra. Both equal to $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ji}$)

$\det A = \det M \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\det M^{-1}}_{(\det M)^{-1}} = -1$.

$MM^{-1} = I$

$\det(M)\underbrace{\det(M^{-1})}_{1/\det M} = \det I = 1$

We must prove (ii) $\Rightarrow$ (iii). If A has order 2 then $A^2 = I$, $A \neq I$. A is a root of $x^2 - 1 = (x+1)(x-1)$

so the minimal poly. of A divides $x^2 - 1$: $m(x) = x^2 - 1$ or $x+1$ or $x-1$ or 1.

If $m(x) = 1$ then $m(A) = I = 0$. No!.

If $m(x) = x - 1$ then $m(A) = A - I = 0$ then $A = I$ (No! I has order 1, not order 2)

If $m(x) = x + 1$ then $m(A) = A + I = 0$ so $A = -I$ (No! by assumption).

So $m(x) = x^2 - 1$ divides $f(x)$, so $f(x) = x^2 - 1$. $\Rightarrow \operatorname{tr} A = 0$, $\det A = -1$. $\Rightarrow$ (i) holds

So ± 1 are eigenvalues of A. Let u, v be eigenvectors corresponding to $1, -1$ i.e. $Au = u$, $Av = -v$.

Let $M = [u \mid v]$ (2×2 matrix having u, v as columns)

$AM = [Au \mid Av] = [u \mid -v] = [u \mid v]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ i.e. (iii) holds. $\square$

There are two conjugacy classes of elements of order 2 in $G = GL_2(\mathbb{R})$:

- $\{-I = \begin{bmatrix}-1&0\\0&-1\end{bmatrix}\}$ is in a class by itself since $-I \in Z(G)$
- All matrices conjugate to $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ i.e. all matrices with trace 0 and determinant -1.
  This includes $\begin{bmatrix}1&a\\0&-1\end{bmatrix}$, $a \in \mathbb{R}$

Consider the dihedral group G of order 8 (the symmetry group of a square), so $|G| = 8$.
Let's pick generators x, y for G where x is an element of order 4 and y is a reflection (order 2).

$G = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$, $\quad yx = x^3y \quad$ i.e. $yxy^{-1} = yxy = x^{-1} = x^3$,

$$\left.\begin{aligned} x^i \cdot x^j &= x^{i+j} \\ x^i \cdot x^jy &= x^{i+j}y \\ x^iy \cdot x^j &= x^{i-j}y \\ x^iy \cdot x^jy &= x^{i-j} \end{aligned}\right\}$$

"If you move y past x^i, it inverts $x^i \mapsto x^{-i}$"

$$x^iyx^jy = x^i\underbrace{(yxy)(yxy)\cdots(yxy)}_{(yxy)^j} = x^i(x^j)^{-1} = x^ix^{-j} = x^{i-j}$$

Presentation for G: $\quad G = \langle \underbrace{x, y}_{\text{generators}} : \underbrace{x^4 = y^2 = 1,\ yx = x^3y}_{\text{relations}} \rangle$

$x^2 \cdot y = x^2y$

$yx^2 = x^{-2}y = x^2y \quad$ $i=0$, $j=2$ in the rule $x^iy \cdot x^j = x^{i-j}y$

g	$\lvert g\rvert$	$C_G(g)$	
1	1	G,	$\lvert G\rvert = 8$
x	4	$\langle x\rangle$	$\lvert\langle x\rangle\rvert = 4$
x^3	4	$\langle x\rangle$	$\lvert\langle x\rangle\rvert = 4$
x^2	2	G,	$\lvert G\rvert = 8$
y	2	$\langle x^2, y\rangle$	$\lvert\langle x^2, y\rangle\rvert = 4$
x^2y	2	$\langle x^2, y\rangle$	$\lvert\langle x^2, y\rangle\rvert = 4$
xy	2	$\langle x^2, xy\rangle$	$\lvert\langle x^2, xy\rangle\rvert = 4$
x^3y	2	$\langle x^2, xy\rangle$	$\lvert\langle x^2, xy\rangle\rvert = 4$

Centralizer of $g \in G$:

$$C_G(g) = \{x \in G : xg = gx\}$$

$Z(G) = \langle x^2\rangle = \{1, x^2\}$.

$C_G(y) = \{1, x^2, y, x^2y\}$ is a Klein four-group

$\mathcal{O}(x) = \{x, x^3\}$

$\mathcal{O}(1) = \{1\}$

$\mathcal{O}(x^2) = \{x^2\}$

$C_G(xy) = \{1, x^2, xy, x^3y\}$ is a Klein four-group

If $\mathcal{O}(g)$ is the conjugacy class of $g \in G$ then $|\mathcal{O}(g)||C_G(g)| = |G|$. $\quad$ eg. $1 \times 8 = 8$, $2 \times 4 = 4$.

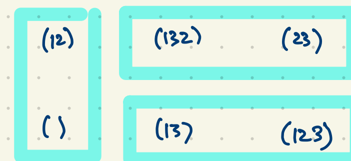
Cosets and Lagrange's Theorem

If H is a subgroup of G (multiplicative, at least generically) then a coset of H in G is a subset of the form $gH = \{gh : h \in H\}$. Note: $gH \subseteq G$, not a subgroup in general.

Eg. take $H = \langle (12) \rangle$ in $G = S_3$. List all cosets of H in G. There are exactly three cosets of H in G: H, $(13)H$, $(23)H$.

$$()H = ()\{(), (12)\} = \{(), (12)\}$$
$$(12)H = (12)\{(), (12)\} = \{(), (12)\}$$
$$(13)H = (13)\{(), (12)\} = \{(13), (123)\}$$
$$(23)H = (23)\{(), (12)\} = \{(23), (132)\}$$
$$(123)H = (123)\{(), (12)\} = \{(123), (13)\}$$
$$(132)H = (132)\{(), (12)\} = \{(132), (23)\}$$

G is partitioned into three cosets, each of size 2.



$$|G| = [G:H]\,|H|$$
$$6 = 3 \times 2$$

(Recall: A partition of G is a collection of subsets that covers all of G without any overlap.)

Theorem The cosets of a subgroup $H \le G$ partition the elements of G.

Proof If $g \in G$, then gH is a coset containing g (since $e \in H$). Suppose two cosets aH and bH overlap i.e. $g \in aH \cap bH$ so $g = ah_1 = bh_2$ for some $h_1, h_2 \in H$, so $aH = gh_1^{-1}H = gH$ and $bH = gh_2^{-1}H = gH$. $\square$
$(a = gh_1^{-1}$ and $b = gh_2^{-1})$

If $h \in H$ then $h = h_1^{-1}h_1h \in h_1^{-1}H$ so $H \subseteq h_1^{-1}H$. Conversely, $h_1^{-1}H \subseteq H$

Theorem All cosets of H in G have cardinality $|gH| = |H|$.

Proof A bijection $H \longrightarrow gH$ is given by $h \longmapsto gh$. An inverse map $gH \to H$ is given by $x \longmapsto g^{-1}x$.

As a corollary, we obtain Lagrange's Theorem: $|G| = \underbrace{(\text{no. of cosets of } H \text{ in } G)}_{\text{the index of } H \text{ in } G \text{ (denoted } [G:H])} \times \underbrace{(\text{size of each coset})}_{|H|}$

i.e. $|G| = [G:H]\,|H|$

Eg. In S_n, the set of all even permutations is a subgroup A_n. $(n \geq 2)$
The set of all odd permutations is a coset of A_n

S_n has two cosets of A_n : $()A_n = A_n = \{\text{even permutations}\}$
$(12)A_n = \{\text{odd permutations}\}$

$$|S_n| = n! = \underbrace{[S_n : A_n]}_{2}\underbrace{|A_n|}_{\frac{n!}{2}}$$

Eg. In the additive group of $\mathbb{R}^3$, a line through the origin is a subgroup.
A coset of this line l is a line parallel to the original line.
The parallel lines to l give a partition of $\mathbb{R}^3$.

Eg. $G = S_n$ is partitioned into cosets of $H = G_1 \cong S_{n-1} = \{\text{permutions of } 2, 3, \dots, n \text{ while fixing } 1\}$
$G = \sigma_1 H \cup \sigma_2 H \cup \sigma_3 H \cup \cdots \cup \sigma_n H$ where $\sigma_k \in G$ is any permutation mapping $1 \mapsto k$ $(k = 1, 2, \dots, n)$.
eg. $\sigma_1 = ()$, $\sigma_2 = (1 2)$, $\sigma_3 = (1 3)$, $\dots$, $\sigma_n = (1 n)$
$\sigma_k H = \{\text{all } \sigma \in G : \sigma(1) = k\}$

Proof If $\sigma \in G$, $\sigma(1) = k$ then $\sigma^{-1}\sigma_k(1) = \sigma^{-1}(k) = 1$ so $\sigma^{-1}\sigma_k \in H = G_1$ so $\sigma^{-1}\sigma_k H = H$ so $\sigma_k H = \sigma H$.

$|H| = (n-1)!$, $[G : H] = n$, $|G| = |H|[G : H]$
$n! = (n-1)! \times n$.

Left cosets vs. Right cosets of $H \leq G$

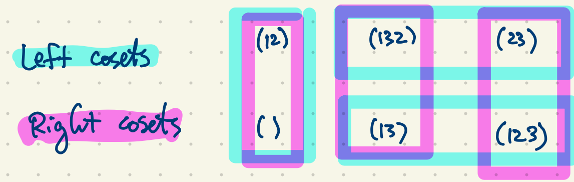
Left cosets $gH = \{gh : h \in H\}$, $g \in G$

Right cosets $Hg = \{hg : h \in H\}$

$[G:H]$ = index of H in G
= number of left cosets of H in G
= number of right cosets of H in G

All cosets of H in G have size $|gH| = |Hg| = |H|$.

Eg. $G = S_3$, $H = S_2 = G_3$



$G_k = \{\sigma \in G : \sigma(k) = k\}$ stabilizer of G

$H = \{(), (12)\}$

$H() = \{(), (12)\}() = \{(), (12)\}$

$H(12) = \{(), (12)\}(12) = \{(12), ()\}$

$H(13) = \{(), (12)\}(13) = \{(13), (132)\}$

$H(23) = \{(), (12)\}(23) = \{(23), (123)\}$

$H(123) = \{(), (12)\}(123) = \{(123), (23)\}$

$H(132) = \{(), (12)\}(132) = \{(132), (13)\}$

If G is abelian, then $gH = Hg$.

We say $H \leq G$ is <u>normal</u> if $gH = Hg$ for all $g \in G$ (left and right cosets are the same).

Eg. $G = S_4$, $K = \langle (12)(34), (13)(24) \rangle = \{(), (12)(34), (13)(24), (14)(23)\}$ is a Klein four-subgroup of G.

<u>Theorem</u> $K \trianglelefteq G$.

<u>Proof</u> If $g \in G$ and $k \in K$ then $gkg^{-1} \in K$ so $gKg^{-1} \subseteq K$. $(gKg^{-1} = \{gkg^{-1} : k \in K\})$.

so $gKg^{-1}g \subseteq Kg$ ie. $gK \subseteq Kg$. Similarly, $gK \supseteq Kg$ so $gK = Kg$. $\square$

In general if $H \leq G$ then gHg^{-1} is a subgroup of G, called a conjugate of H. (conjugating by $g \in G$)

<u>Proof</u> Given $h_1, h_2 \in H$, so $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$, we have $(gh_1g^{-1})(gh_2g^{-1}) = g\underline{h_1h_2}g^{-1} \in gHg^{-1}$. Take $e \in G$ as the identity, so $e \in H$ and $geg^{-1} = e \in gHg^{-1}$. Also if $h \in H$, so $ghg^{-1} \in gHg^{-1}$, then $(ghg^{-1})^{-1} = gh^{-1}g^{-1} \in gHg^{-1}$.

Conjugate subgroups are isomorphic to each other. Given $g \in G$, $H \leq G$, an isomorphism $H \longrightarrow gHg^{-1}$ is given by $h \mapsto ghg^{-1}$.

A subgroup $H \leq G$ is <u>normal</u> ($H \trianglelefteq G$) iff every conjugate of H is H itself i.e. $gHg^{-1} = H$ for all $g \in G$.

<u>Example</u> $G = S_4$, $H = G_1 = \{(), (23), (24), (34), (234), (243)\} \cong S_3$, $g = (124) \notin H$.

$g^{-1} = (142)$

$gHg^{-1} = G_2 = \{(), (13), (14), (34), (134), (143)\} \cong S_3$

$= \langle (13), (14) \rangle$

why? Given $h \in H = G_1$, $ghg^{-1}(2) = gh(1) = g(1) = 2$. so $ghg^{-1} \in G_2$. This shows $gHg^{-1} \subseteq G_2$.

In fact $gHg^{-1} = G_2$.

<u>Theorem</u> Every conjugacy class in G has size (cardinality) dividing $|G|$.

Eg. A_4 has four conjugacy classes $\{()\}$, $\{(12)(34), (13)(24), (14)(23)\}$, $\{(124), (132), (143), (234)\}$, $\{(142), (123), (134), (243)\}$.

$$(123)(12)(34)\underbrace{(123)^{-1}}_{(132)} = (23)(14) = (14)(23), \qquad (132)(12)(34)(132)^{-1} = (31)(24) = (13)(24).$$

$$(123)(124)(123)^{-1} = (234)$$

In S_4, (124) is conjugate to (142) since they have the same cycle structure:

$$(24)(124)(24)^{-1} = (142)$$
$$(14)(124)(14)^{-1} = (421)$$

Eg. <u>Theorem</u> A_4 has no subgroup of order 6.

<u>Proof</u> Suppose $G = A_4$ has a normal subgroup $K \trianglelefteq G$ of order $|K| = 6$. Partitioning G into left cosets $G = K \cup gK$ where $g \notin K$ $\left([G:K] = \frac{|G|}{|K|} = \frac{12}{6} = 2 \right)$ and partition G into right cosets as $G = K \cup Kg$ so $gK = Kg$. So $gKg^{-1} = K$.

Let G, H be groups (assumed to be multiplicative with identity elements $e_G \in G$, $e_H \in H$).

A <u>homomorphism</u> $G \to H$ is a map satisfying $\phi(gg') = \phi(g)\phi(g')$ for all $g, g' \in G$.

Note: An isomorphism is the same thing as a bijective homomorphism.

<u>Eg.</u> $\phi: \underbrace{GL_n(F)}_{\substack{\text{invertible} \\ n\times n \text{ matrices} \\ \text{over a field } F}} \to \underbrace{F^\times}_{\substack{\text{multiplicative} \\ \text{group of nonzero} \\ \text{elements of } F}}$, $\phi = \det$.

Properties: $\phi(e_G) = e_H$. $\quad (\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G) \Rightarrow \phi(e_G) = e_H)$.

If $g \in G$ has order n then $|\phi(g)|$ divides $n = |g|$. $\quad$ eg. if $|g| = 6$ then $|\phi(g)|$ has order 1, 2, 3 or 6.

$g^n = e_G \Rightarrow \underbrace{\phi(g^n)}_{=\phi(g)^n} = \phi(e_G) = e_H$

$\phi(g^{-1}) = \phi(g)^{-1}$ since $gg^{-1} = e_G \Rightarrow \underbrace{\phi(gg^{-1})}_{=\phi(g)\phi(g^{-1})} = \phi(e_G) = e_H$

The <u>kernel</u> of a homomorphism $\phi: G \to H$ is $\ker\phi = \{g \in G : \phi(g) = e_H\}$. (Compare: the null space of a linear transformation)

<u>Theorem</u>: $\ker\phi$ is a subgroup of G.

<u>Proof</u> If $g, g' \in \ker\phi$ then $\phi(g) = \phi(g') = e_G$ then $\phi(gg') = \phi(g)\phi(g') = e_G e_G = e_G$ so $gg' \in \ker\phi$.

Since $\phi(e_G) = e_H$, $e_G \in \ker\phi$.

If $g \in \ker\phi$ then $\phi(g) = e_H$ so $\phi(g^{-1}) = \phi(g)^{-1} = e_H^{-1} = e_H$ so $g^{-1} \in \ker\phi$. $\quad$ So $\ker\phi \leq G$.

Note: If ϕ is one-to-one then $\ker\phi = \{e_G\}$. Conversely, if $\ker\phi = \{e_G\}$ then we show ϕ is one-to-one:

If $\phi(g) = \phi(g')$ then $\phi(g^{-1}g') = \phi(g^{-1})\phi(g') = \phi(g)^{-1}\phi(g') = e_H$ i.e. $g^{-1}g' \in \ker\phi = \{e_G\}$ so $g^{-1}g' = e_G$ so $g' = g$. $\quad \square$

The image of a homomorphism $\phi: G \to H$ then the image $\phi(G) = \{\phi(g) : g \in G\}$ is a subgroup of H.

Proof Given two elements in $\phi(G)$, say $\phi(g), \phi(g')$ for some $g, g' \in G$, then $\phi(g)\phi(g') = \phi(gg') \in \phi(G)$. Also $e_H = \phi(e_G) \in \phi(G)$. If we take any element in $\phi(G)$, say $\phi(g)$ where $g \in G$, then $\phi(g)^{-1} = \phi(g^{-1}) \in \phi(G)$. So $\phi(G) \le H$. $\square$

Note: $\phi: G \to H$ is onto iff $\phi(G) = H$.

Eg. Define $\phi: S_4 \to S_3$ as follows: Take $\pi_1 = (12)(34)$, $\pi_2 = (13)(24)$, $\pi_3 = (14)(23)$ in S_4. These form a conjugacy class in S_4 $\{\pi_1, \pi_2, \pi_3\} = X$. (Really $\phi(\sigma) \in \text{Sym}\, X = \text{Sym}\{\pi_1, \pi_2, \pi_3\}$).

Given $\sigma \in S_4$, we have a map $X \to X$, $\pi_i \xmapsto{\phi(\sigma)} \sigma \pi_i \sigma^{-1}$.

Eg. $\phi((13))$:
$$\begin{aligned}
\pi_1 &\mapsto (13)\pi_1(13)^{-1} = (13)(12)(34)(13)^{-1} = (32)(14) = (14)(23) = \pi_3 \\
\pi_2 &\mapsto (13)\pi_2(13)^{-1} = (13)(13)(24)(13)^{-1} = (31)(24) = (13)(24) = \pi_2 \\
\pi_3 &\mapsto (13)\pi_3(13)^{-1} = (13)(14)(23)(13)^{-1} = (34)(21) = (12)(34) = \pi_1
\end{aligned}$$
$\phi((13)) = (13)$

$\phi((142))$:
$$\begin{aligned}
\pi_1 &\mapsto (142)\pi_1(142)^{-1} = (142)(12)(34)(142)^{-1} = (41)(32) = (14)(23) = \pi_3 \\
\pi_2 &\mapsto (142)\pi_2(142)^{-1} = (142)(13)(24)(142)^{-1} = (43)(12) = (12)(34) = \pi_1 \\
\pi_3 &\mapsto (142)\pi_3(142)^{-1} = (142)(14)(23)(142)^{-1} = (42)(13) = (13)(24) = \pi_2
\end{aligned}$$
$\phi((142)) = (132)$

ϕ is onto S_3. (why? $\phi(S_4)$ is a subgroup of S_3. By Lagrange's Theorem, $|\phi(S_4)|$ is divisible by $|\phi((13))| = |(13)| = 2$ and $|\phi((142))| = |(132)| = 3$ so $\phi(S_4) = S_3$.)

$\ker\phi = C_{S_4}(X) = \langle \pi_1, \pi_2 \rangle = \{(), \pi_1, \pi_2, \pi_3\}$ is a Klein four subgroup of order 4 in S_4.
$(\pi_3 = \pi_1\pi_2)$

ϕ is a homomorphism; it is 4-to-1.

The image of a homomorphism $\phi: G \to H$ i.e. the subgroup $\phi(G) = \{\phi(g) : g \in G\} \le H$ is a <u>homomorphic image</u> of G.

Fractional Linear Transformations (or Linear Fractional Transformations)

A map $\mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ (actually a permutation) of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \longmapsto \dfrac{ax+b}{cx+d}$ where $ad-bc \neq 0$.

$GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad-bc \neq 0 \right\}$ for actual invertible 2×2 real matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}(x) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right) = \frac{a\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)+b}{c\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)+d} = \frac{a(\alpha x+\beta)+b(\gamma x+\delta)}{c(\alpha x+\beta)+d(\gamma x+\delta)} = \frac{(a\alpha+b\gamma)x+(a\beta+b\delta)}{(c\alpha+d\gamma)x+(c\beta+d\delta)}$$

$$= \begin{bmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{bmatrix}(x)$$

Compare with multiplication of actual 2×2 invertible matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix}$$

We denote by $PGL_2(\mathbb{R})$ the group of all fractional linear transformations $\mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ i.e.

$$PGL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}.$$

This is a homomorphic image of $GL_2(\mathbb{R})$ under the homomorphism $\phi : GL_2(\mathbb{R}) \to PGL_2(\mathbb{R})$,
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. This map is a homomorphism: $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) = \phi\left(\begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix}\right)$

$$= \begin{bmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\phi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right).$$

This homomorphism is onto $PGL_2(\mathbb{R})$ by definition but it's not onto because $\phi\left(\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}\right) = \begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

since $\begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}(x) = \dfrac{\lambda a x+\lambda b}{\lambda c x+\lambda d} = \dfrac{ax+b}{cx+d} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}(x)$

$$\begin{bmatrix}3&4\\1&7\end{bmatrix}(5) = \frac{3\times5+4}{1\times5+7} = \frac{19}{12}.$$

$$\begin{bmatrix}3&4\\1&7\end{bmatrix}(\infty) = \frac{3\times\infty+4}{1\times\infty+7} = 3$$

$$\begin{bmatrix}3&4\\1&7\end{bmatrix}(-7) = \frac{3\times(-7)+4}{1\times(-7)+7} = \frac{-17}{0} = \infty$$

$$\begin{bmatrix}3&4\\0&7\end{bmatrix}(\infty) = \frac{3\times\infty+4}{\cancel{0\times\infty}+7} = \infty.$$

In $GL_2(\mathbb{R})$, $\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$. $(ad-bc\neq 0)$

$\mathbb{F}_q$ = field of order q

$|GL_2(\mathbb{F}_q)| = (q^2-1)(q^2-q)$

$|SL_2(\mathbb{F}_q)| = (q^2-1)q$

divide by $q-1$.

Every fractional linear transformation is a permutation of $\mathbb{R}\cup\{\infty\}$

$PGL_2(\mathbb{R})$ is a group. $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix} = \begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$

The identity $\begin{bmatrix}1&0\\0&1\end{bmatrix}(x) = \frac{1\times x \cancel{+0}}{\cancel{0\times x}+1} = x.$

You can think of $PGL_2(\mathbb{R})$ as the same as 2×2 invertible matrices but where we identify nonzero scalar multiples i.e. $\lambda\begin{bmatrix}a&b\\c&d\end{bmatrix} = \begin{bmatrix}\lambda a&\lambda b\\\lambda c&\lambda d\end{bmatrix} = \begin{bmatrix}a&b\\c&d\end{bmatrix}.$

$GL_2(\mathbb{F}_2) = \{\overline{\begin{pmatrix}1&0\\0&1\end{pmatrix}}, \overline{\begin{pmatrix}1&1\\0&1\end{pmatrix}}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix}, \begin{pmatrix}1&0\\1&1\end{pmatrix}\}. = SL_2(\mathbb{F}_2).$

$|GL_2(\mathbb{F}_2)| = (2^2-1)(2^2-2) = 3\times2 = 6.$

$\mathbb{F}_2 = \{0,1\}$ is the field of order 2:

$PGL_2(\mathbb{F}_2) = \{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}\} \cong GL_2(\mathbb{F}_q) \cong SL_2(\mathbb{F}_2) \cong S_3$

why? $PGL_2(\mathbb{F}_2)$ is a group of permutations of $\{\underbrace{0,1}_{\mathbb{F}_2},\infty\}$

$\mathrm{Sym}\{0,1,\infty\} = \{$all permutations of $0,1,\infty\}$

so $PGL_2(\mathbb{F}_2)$ is isomorphic to a subgroup of S_3.

$|GL_2(\mathbb{F}_3)| = (3^2-1)(3^2-3) = 8\times6 = 48$

$\mathbb{F}_3 = \{0,1,2\}$ $\quad \frac{1}{2} = 2 = -1$ $\quad |PGL_2(\mathbb{F}_3)| = \frac{48}{2} = 24$ $\quad PGL_2(\mathbb{F}_3) \cong S_4.$

The map $GL_2(\mathbb{F}_3) \to PGL_2(\mathbb{F}_3)$ is 2-to-1.

$PGL_2(\mathbb{F}_3)$ is a group of permutations of $\mathbb{F}_3\cup\{\infty\} = \{0,1,2,\infty\}.$

$\begin{pmatrix}a&b\\c&d\end{pmatrix} \mapsto \begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$ field of order 4

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

$|GL_2(\mathbb{F}_4)| = (4^2-1)(4^2-4) = 15 \times 12 = 180$

$|SL_2(\mathbb{F}_4)| = \frac{180}{3} = 60$

$|A_5| = \frac{5!}{2} = 60$

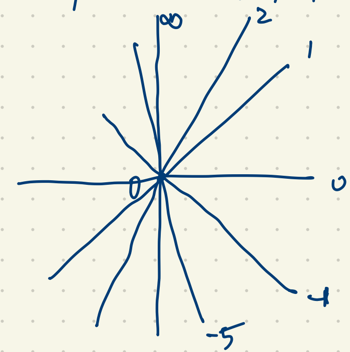
$SL_2(\mathbb{F}_4) \cong A_5$.

$PSL_2(\mathbb{F}_4) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc = 1, \quad a,b,c,d \in \mathbb{F}_4 \right\} \cong SL_2(\mathbb{F}_4)$

The map $SL_2(\mathbb{F}_4) \longrightarrow PSL_2(\mathbb{F}_4)$ acting as all even permutations of $\mathbb{F}_4 \cup \{\infty\} = \{0, 1, \alpha, \beta, \infty\}$

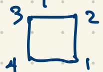
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}(x) = \frac{1 \times x + 1}{0 \times x + 1} = x + 1 \quad : \quad (0, 1)(\alpha, \beta)(\infty)$



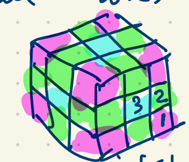
$\mathbb{R} \cup \{\infty\} = \{$all possible slopes of lines through the origin in $\mathbb{R}^2\}$

Orbits and Stabilizers for Group Actions

Eg. $G =$ symmetry group of 

, $G < S_4$, $G = \langle (1234), (13) \rangle$ a dihedral group of order 8.
G permutes the four vertices transitively (meaning if $x, y \in \{1,2,3,4\}$ then there exists $g \in G$ such that $g(x) = y$).

For legal moves of a Rubik's cube, the group of all moves does not permute the 26 small cubes (the group has three orbits of size 12, 8, 6)
$12 + 8 + 6 = 26$.



$\mathcal{O}(1) = \{$all small corner cubes$\}$, $|\mathcal{O}(1)| = 8$
$|\mathcal{O}(2)| = 12$
$|\mathcal{O}(3)| = 6$

A group action is transitive if there is only only one orbit.

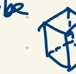
The stabilizer of x is $\mathrm{Stab}_G(x) = G_x = \{ g \in G : g(x) = x \} \leq G$. (a subgroup)

eg. in the dihedral group above, $\mathrm{Stab}_G(2) = G_2 = \{$all elements of G fixing $2\} = \{(), (1\ 3)\} = \langle (13) \rangle$
$\mathrm{Stab}_G(1) = \{(), (24)\} = \mathrm{Stab}_G(3) = \langle (24) \rangle$

The orbit of x is $\mathcal{O}(x) = \{ g(x) : g \in G \}$. In this case there is only one orbit
$\mathcal{O}(1) = \{1, 2, 3, 4\} = \mathcal{O}(2) = \mathcal{O}(3) = \mathcal{O}(4)$

Theorem If G permutes $X = [n] = \{1, 2, \dots, n\}$ then for every $x \in X$, $|\mathrm{Stab}_G(x)|\,|\mathcal{O}(x)| = |G|$.

In our dihedral group of order 8:
$|\mathrm{Stab}_G(x)| = 2$, $|\mathcal{O}(x)| = 4$, $|G| = 8$

We have implicitly used this! eg. when calculating the symmetry group G of a cube 

$|G| = |Stab(v)||O(v)|$ where v is a vertex

$= 6 \times 8 = 48$

or

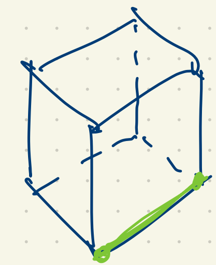
$|G| = |Stab(F)||O(F)|$ where F is a face

$= 8 \times 6 = 48$

or

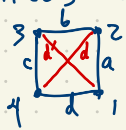
$|G| = |Stab(e)||O(e)|$

$= 4 \times 12 = 48$



More examples of stabilizers and orbits

$G = \langle (1234), (13) \rangle$



G also permutes the four edges a, b, c, d transitively

$Stab_G(a) = \langle (12)(34) \rangle = \{(), (12)(34)\}$

$O(a) = \{a, b, c, d\}$

$|G| = |Stab(a)||O(a)|$

$8 = 2 \times 4$

G also permutes the two diagonals d, d'

$O(d) = \{d, d'\}$

$Stab(d) = \{(), (13), (24), (13)(24)\}$, a Klein four-group

$|G| = \underline{|Stab(d)|}\,\underline{|O(d)|}$

$8 = 4 \times 2$

$Stab_G(x) \leq G$ is a subgroup

$O(x) \subseteq X$ is not a group, just a set of points

$G = GL_3(F)$ where F is a field

G acts on F^3, permuting vectors

The stabilizer of $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ is $\mathrm{Stab}_G(e_1) = \{ g \in G : ge_1 = e_1 \}$

$ge_1 = e_1$ says $\begin{bmatrix}1 & b & c\\ 0 & e & f\\ 0 & i & j\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

$$= \left\{ \begin{bmatrix}1 & b & c\\ 0 & e & f\\ 0 & i & j\end{bmatrix} : b,c,e,f,i,j \in F,\ ej - fi \neq 0 \right\}$$

$$\mathcal{O}(e_1) = \{\text{all nonzero vectors}\} = F^3 - \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

F^3 has two orbits: $\Big\{\underbrace{\begin{bmatrix}0\\0\\0\end{bmatrix}}_{0}\Big\}$, $F^3 - \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$.

$\mathrm{Stab}_G(0) = G$

<u>Theorem</u> If G acts on X (i.e. G permutes X i.e. $G \leq \underset{(\mathrm{Perm}\,X)}{\mathrm{Sym}\,X}$) and $x \in X$ (any point) then $|\mathrm{Stab}_G(x)| \cdot |\mathcal{O}(x)| = |G|$.

<u>Proof</u> Let $H = \mathrm{Stab}_G(x)$ and $\mathcal{O}(x) = \{\underset{x}{\overset{\parallel}{x_1}}, x_2, \dots, x_k\} \subseteq X$. Then there exist $g_1, \dots, g_k \in G$ such that $g_i(x) = x_i$ (by definition). (Note: $g_1, \dots, g_k$ are not uniquely determined.)

Then $G = g_1H \sqcup g_2H \sqcup g_3H \sqcup \dots \sqcup g_kH$. ($A \sqcup B$ denotes disjoint union i.e. $A \cup B$ with no overlap, $A \cap B = \emptyset$)

Why? If $g \in G$ then $g(x) \in \mathcal{O}(x)$ so $g(x) = x_i$ for ~~some~~ $i \in \{1, 2, \dots, k\}$ and $g_i(x) = x_i$ so $g_i^{-1}(g(x)) = g_i^{-1}(x_i) = x$ so $g_i^{-1}g \in H = \mathrm{Stab}(x)$

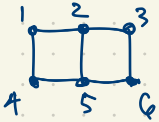
so $g_i^{-1}gH = H$ i.e. $\underset{g \in}{} gH = g_iH$.

In fact $g_iH = \{g \in G : g(x) = x_i\}$.

Now $k = |\mathcal{O}(x)| = [G : H]$ and

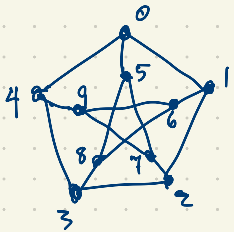
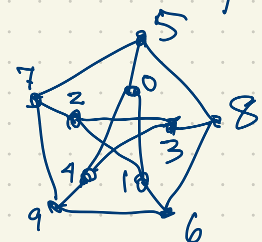
$|G| = |H|[G:H] = |\mathrm{Stab}(x)||\mathcal{O}(x)|$.

Application to graph theory : computing the number of automorphisms of a graph.

Eg. $\Gamma =$  has four automorphisms. Its automorphism group is a Klein four-group

$G = \langle (13)(46), \; (14)(25)(36) \rangle = \{ (), (13)(46), (14)(25)(36), (16)(25)(34) \}$

G has two orbits on vertices: $\{1,3,4,6\}$, $\{2,5\}$.

Eg. $P =$  $=$  has automorphisms including $(0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)$,

$(0\ 5)(1\ 8\ 4\ 7)(2\ 6\ 3\ 9)$

$(0\ 5)(1\ 7\ 4\ 8)(2\ 9\ 3\ 6)$

P is the Petersen graph

How many automorphisms does P have?

$\mathrm{Aut}\, P = \{ \text{automorphisms of } P \} \leq S_{10}$ actually $\mathrm{Sym}\{0,1,2,\dots,9\}$

Theorem $|\mathrm{Aut}\, P| = 120$. Is $\mathrm{Aut}\, P \cong S_5$?

Proof First enumerate orbits of $G = \mathrm{Aut}\, P$ on the vertex set $\{0,1,2,\dots,9\}$.

There is only one orbit by considering the dihedral subgroup of order 10 and $(0\ 5)(1\ 8\ 4\ 7)(2\ 6\ 3\ 9)$. So G is transitive on vertices $|G| = 10\,|G_0|$. where $G_0 = \mathrm{Stab}_G(0)$.