

Math 3500

Algebra I: Group Theory

Book 1

Symmetry group of a square  :

$$G = \{I, R, R^2, R^3, H, V, D, D'\}$$

R = counter-clockwise rotation about center by 90°

R^2 = 180° rotation about center

R^3 = 270° counterclockwise rotation = 90° clockwise rotation

$$R^4 = I$$

D = reflection



$$H = \text{---} \square \text{---} \updownarrow H$$

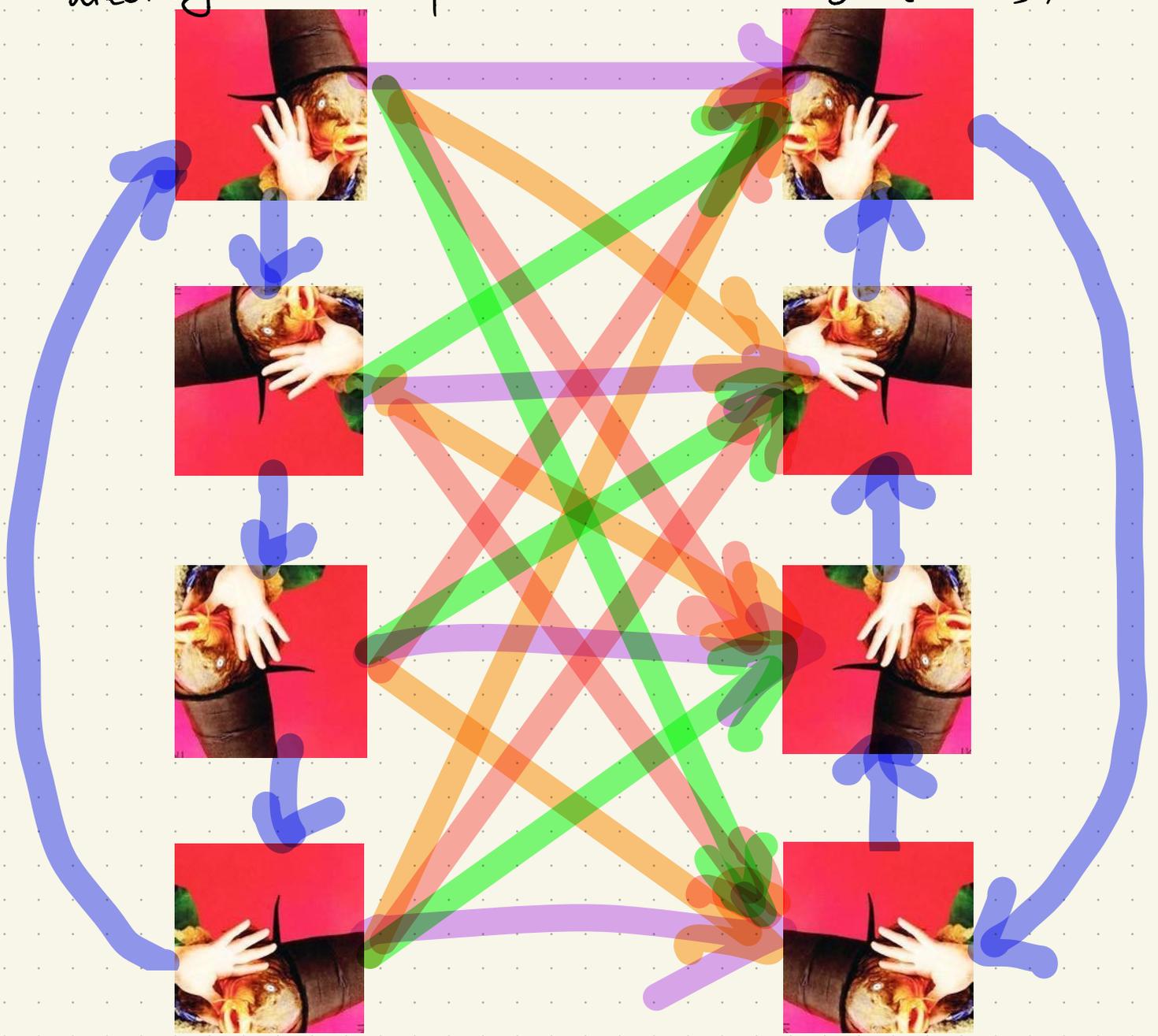
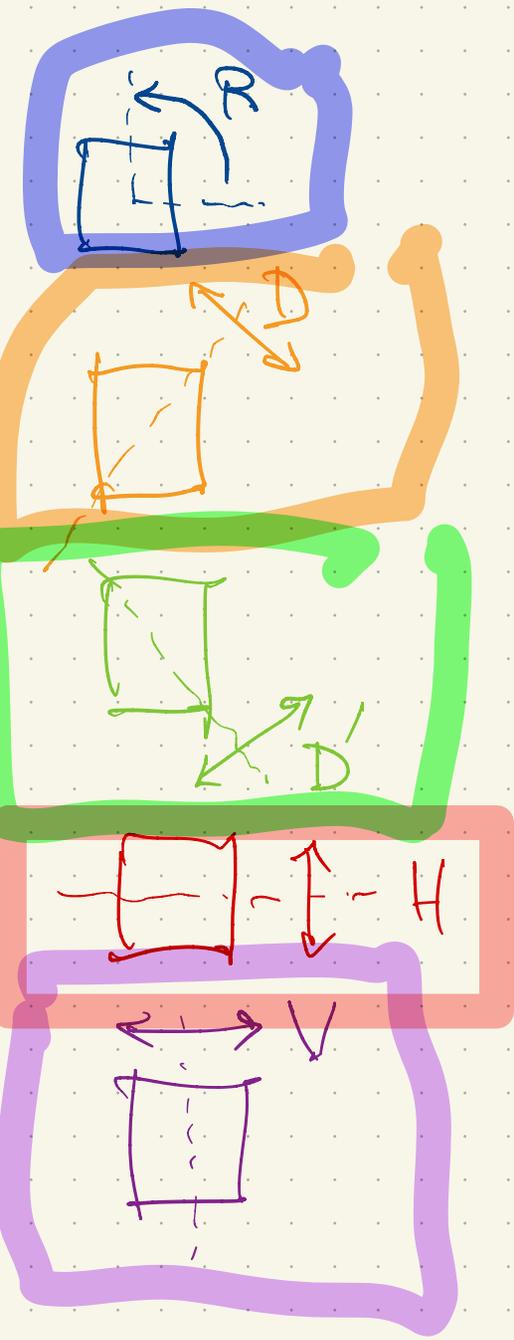
$$V = \square \text{---} \leftarrow \rightarrow V$$

$$D' = \text{---} \square \text{---} \nearrow \searrow$$



Symmetry group of square $G = \{I, R, R^2, R^3, D, D', H, V\}$

Group elements are transformations/functions/maps/mappings/arrows (not the images/squares on which the group elements act).
 Virtual symmetries reverse orientation; (eg. reflections)
 direct symmetries preserve orientation. (eg. rotations)

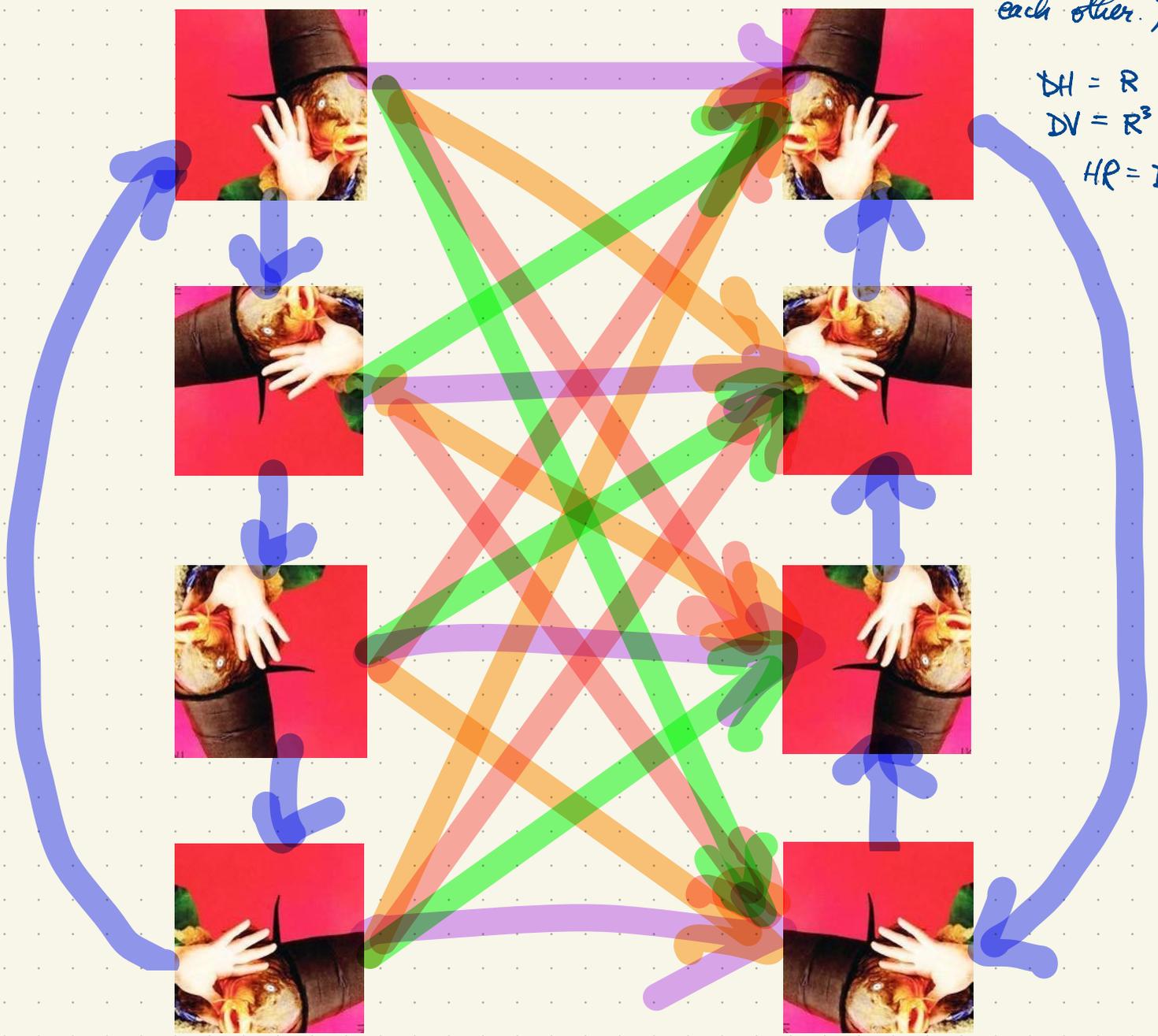
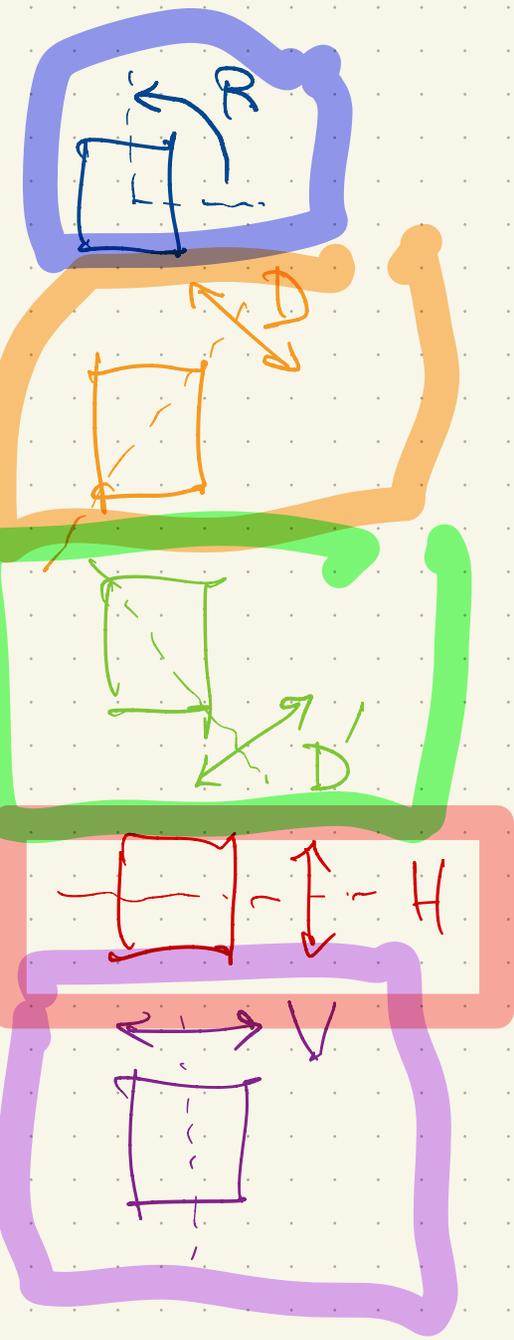


Symmetry group of square $G = \{I, R, R^2, R^3, D, D', H, V\}$

Composition (right-to-left)
 $RD = V$
 $DR = H$
 $HV = R^2$
 $VH = R^2$

We say that G is non-abelian because its elements do not all commute with each other. (A group is abelian iff all its elements commute with each other.)

Note: H and V commute (ie. $HV = VH$) but R and D do not commute ($RD \neq DR$)



$DH = R$
 $DV = R^3$
 $HR = D'$

The multiplication table of G :

	I	R	R^2	R^3	D	D'	H	V
I	I	R	R^2	R^3	D	D'	H	V
R	R	R^2	R^3	I	V	H	D	D'
R^2	R^2	R^3	I	R	D'	D	V	H
R^3	R^3	I	R	R^2	H	V	D'	D
D	D	H	D'	V	I	R^2	R	R^3
D'	D'	V	D	H	R^2	I	R^3	R
H	H	D'	V	D	R^3	R	I	R^2
V	V	D	H	D'	R	R^3	R^2	I

$$DR^2 = DR \cdot R = HR = D'$$

$$D'R^2 = D'R \cdot R = VR = D$$

The (i,j) entry (i.e. row i , column j) indicates the i^{th} element "times" the j^{th} element.

In the multiplication table, each group element appears exactly once in each row and column.

	T
S	U
W	U

$$\Rightarrow ST = U = WT \Rightarrow ST \cdot T^{-1} = WT \cdot T^{-1} \Rightarrow S = W$$

Associativity holds!
 $f \circ (g \circ h) = (f \circ g) \circ h$
 $f(g(h(x)))$