The background of the entire image is a dense, repeating geometric pattern. It consists of interlocking shapes in three colors: red, blue, and gold. The red shapes are triangles with internal patterns, the blue shapes are hexagons with internal patterns, and the gold shapes are star-like or floral motifs. These shapes are arranged in a grid-like fashion, creating a complex, tessellated effect. The overall appearance is reminiscent of a traditional Islamic or Arabesque geometric design.

Math 3500

Algebra I: Group Theory

Book 2

Similar to HW#2: How many elements of each order does S_4 have?

1 element of order 1: $() = \text{identity}$

9 elements of order 2: $(12), (13), (14), (23), (24), (34),$ $\leftarrow \binom{4}{2} = 6$ transpositions
 $(12)(34), (13)(24), (14)(23)$ $\leftarrow \frac{1}{2!} \binom{4}{2} \binom{2}{2} = \frac{6}{2} = 3$

8 elements of order 3: $(123), (124), (132), (134), (142), (143), (234), (243)$

6 elements of order 4: $(1234), (1243), (1324), (1342), (1423), (1432)$

$24 = 4! = |S_4|$ $|S_n| = n! = 1 \times 2 \times 3 \times \dots \times n$

In S_n the number of n -cycles is $(n-1)!$

The binomial coefficient $\binom{n}{k}$ ("n choose k") is the number of ways to choose a subset of size k from a set of size n .

$\binom{n}{k} = k^{\text{th}}$ entry in n^{th} row of Pascal's Triangle

$\binom{4}{2} = \text{number of 2-subsets in } [4] = \{1,2,3,4\}$
 $= 6$

By the way, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Theorem $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Pascal's Triangle

$n=0$	1							
$n=1$	1	1						
$n=2$	1	2	1					
$n=3$	1	3	3	1				
$n=4$	1	4	6	4	1			
$n=5$	1	5	10	10	5	1		
$n=6$	1	6	15	20	15	6	1	
$n=7$	1	7	21	35	35	21	7	1

A transposition is a 2-cycle $(i j) \in S_n$, $i \neq j$ in $[n] = \{1, 2, \dots, n\}$.

Products of disjoint transpositions eg. $(1 3)(2 5)(6 8) \in S_8$
are elements of order 2.

How many elements of order 2 are there in S_7 ?

Transpositions: $(12), (13), (14), \dots, (67)$ i.e. $(i j)$ where $i \neq j$ in $[7] = \{1, 2, \dots, 7\}$

$$\binom{7}{2} = 21 \text{ transpositions}$$

Products of two disjoint transpositions eg. $(26)(34) = (34)(26)$

$$\text{Number of these is } \frac{1}{2} \binom{7}{2} \binom{5}{2} = 105$$

Products of three disjoint transpositions eg. $(15)(27)(36) = (15)(36)(27) = (27)(36)(15)$

$$\text{Number of these is } \frac{1}{6} \binom{7}{2} \binom{5}{2} \binom{3}{2} = \frac{21 \times 10 \times 3}{6 \cdot 2} = 105$$

Number of 3-cycles in S_7 eg. (274) : $2 \binom{7}{3} = 2 \times 35 = 70$

Number of products of two disjoint 3-cycles: eg. $(274)(356) = (356)(274)$

$$\frac{1}{2} \cdot 70 \cdot \binom{4}{3} \cdot 2 = 70 \cdot 4 = 280$$

Elements of order 12 in S_7 eg. $(142)(3756)$

$$70 \cdot 3! = 70 \cdot 6 = 420 \text{ elements of order 12 in } S_7$$

$$280 + 70$$

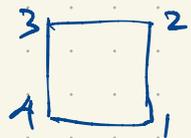
$$= 350$$

elements
of order 3
in S_7

Revisiting the dihedral group of order 8 (symmetry group of a square)

$$G = \{ I, R, R^2, R^3, D, D', V, H \}$$

viewed as a group of permutations of the four vertices



$$I \mapsto ()$$

$$R \mapsto (1234)$$

$$R^2 \mapsto (13)(24)$$

$$R^3 \mapsto (1432)$$

$$H \mapsto (12)(34)$$

$$V \mapsto (14)(23)$$

$$D \mapsto (13)$$

$$D' \mapsto (24)$$

$G \cong$ subgroup of S_4 : $\{(), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$

$$RV = (1234)(14)(23) = (24) = D' = \langle (1234), (13) \rangle$$

This is an example of a permutation group of degree 4, i.e. a subgroup of S_4 .
A permutation group of degree n is a subgroup of the symmetric group S_n .

Theorem (Cayley's Representation Theorem) Every finite group is isomorphic to a permutation group. In fact, if $|G| = n$ then G is isomorphic to a subgroup of S_n . (But we can usually do better. Eg. the dihedral group of order 8 is isomorphic to a subgroup of S_8 . But S_4 is even better.)