Algebra I

Group Theory

Book 2

Transpositions (i j) are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd; and vice versa.
A cycle of odd length is an even permutation;
" " " even " " " odd " .

If α is a product of an even number of transpositions, then α is an even permutation.
" " " " " " odd " " , " " " " odd " .

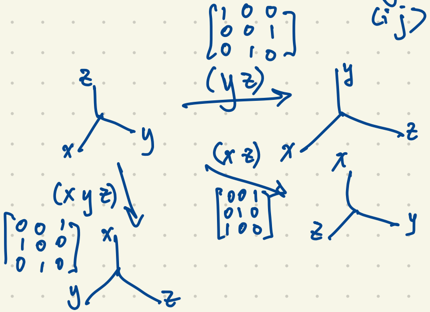
Permutations in S_5:

Even		Odd	
$()$	1	$(i\,j)$	10
$(i\,j\,k)$	20	$(i\,j\,k\,l)$	30
$(i\,j\,k\,l\,m)$	24	$(i\,j\,k)(l\,m)$	20
$(i\,j)(k\,l)$	15		
	60		60

$|S_5| = 120$

$A_5 = \{$ even permutations in $S_5 \}$

$|A_5| = 60$



An even permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-preserving transformation.

An odd permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-reversing transformation.

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then

$$\det T \begin{cases} = 0 & \text{if } T \text{ is not invertible} \\ > 0 & \text{" " preserves orientation} \\ < 0 & \text{" " reverses " "} \end{cases}$$

A permutation $\alpha \in S_n$ can be expressed as a product of transpositions.

If α is a product of an even number of transpositions, then α is even.

- - - - - . . odd - - - - - - - - - - . odd.

In S_3:

$(1\,3)(1\,2)(1\,3)(2\,3)(2\,3)(2\,3)(1\,2)(2\,3) = (1\;2\;3)$ says $(1\,2\,3)$ is an even permutation.

$S_3 \cong \left\langle \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\-1&-1\end{bmatrix} \right\rangle \cong$ dihedral group of order 6 (symmetry group of an equilateral triangle)

n	no. of groups of order n up to isomorphis
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	5

Groups of order 2

$S_2 \cong \{0, 1\}$ mod 2 under addition $\cong \langle -1 \rangle$ under multiplication

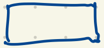
$\circ$	$()$	(12)
$()$	$()$	(12)
(12)	(12)	$()$

+	0	1
0	0	1
1	1	0

$\cdot$	1	−1
1	1	−1
−1	−1	1

Cayley tables of groups of order 2 all "look the same"

has a cyclic symmetry group of order 4



has an abelian symmetry group of order 4 which is not cyclic (the Klein four-group)

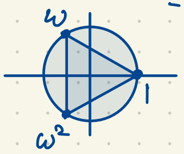
Theorem Any two groups of prime order are isomorphic; they are cyclic of order p.

Eg. $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ (under addition mod 3) is ~~isomorplic~~ isomorphic to $A_3 = \langle (123) \rangle = \{(), (123), (132)\}$ and $\{1, \omega, \omega^2\}$ under multiplication, $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∘	()	(123)	(132)
()	()	(123)	(132)
(123)	(123)	(132)	()
(132)	(132)	()	(123)

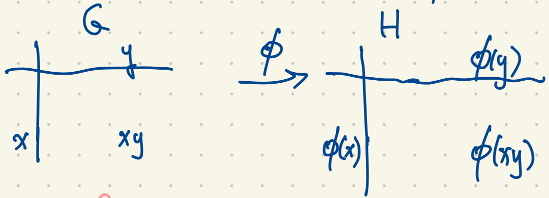
·	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω



We say two groups G, H are <u>isomorphic</u> $(G \cong H)$ if there exists a bijection $\phi : G \to H$ such that $\phi(xy) = \phi(x)\phi(y)$

operation in G

operation in H



An isomorphism $\phi : \mathbb{Z}/3\mathbb{Z} \to A_3$ is a bijection satisfying $\phi(x+y) = \phi(x) \circ \phi(y)$

An isomorphism $\phi : \mathbb{R} \longrightarrow (0,\infty)$, $\phi(x+y) = \phi(x)\phi(y)$ is defined by $\phi(x) = e^x$

$\mathbb{R}$ under addition

$(0,\infty)$ under multiplication (subgroup of $\mathbb{R}^\times = (-\infty,0) \cup (0,\infty)$)

$e^{x+y} = e^x \cdot e^y$.

$\ln = \phi^{-1} : (0,\infty) \longrightarrow \mathbb{R}$

$\mathbb{R} \not\cong \mathbb{R}^\times$

since $\mathbb{R}$ (reals under addition) has only one element of finite order whereas $\mathbb{R}^\times$ has two elements of finite order : ± 1.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}/3\mathbb{Z}$

is isomorphic to

*	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\phi(0) = c$
$\phi(1) = a$
$\phi(2) = b$

*	c	a	b
c	c	a	b
a	a	b	c
b	b	c	a

or

$\phi(0) = c$
$\phi(1) = b$
$\phi(2) = a$

*	c	b	a
c	c	b	a
b	b	a	c
a	a	c	b

+	0
0	0

Every group of order 1 is isomorphic to

·	1
1	1

(trivial group $\{1\}$)

〃 · · — — — · · 2 〃 — — · ·· $\mathbb{Z}/2\mathbb{Z}$

+	0	1
0	0	1
1	1	0

	c
a	ac
b	bc

If $ac=bc$ then multiply both sides by c^{-1} on the right to get

$$(ac)c^{-1}=(bc)c^{-1}$$
$$a(cc^{-1}) = b(cc^{-1})$$
$$a1 = b1$$
$$a = b$$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Every group of order 3 is cyclic (isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under addition).

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Two cases: either all non-identity elements ~~of~~ G have order 2, or G has an element not of order 2.

Theorem: There are exactly two groups of order 4 up to isomorphism: the Klein four-group and the cyclic group of order 4.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

cyclic group of order 5

$$\langle a\rangle = \{e, a, \underset{\underset{b}{\uparrow}}{a^2}, \underset{\underset{c}{\uparrow}}{a^3}, \underset{\underset{d}{\uparrow}}{a^4}\}$$

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	a	e
c	c	d	e	b	a
d	d	b	a	e	c

is not a group!

It is a quasigroup, in fact since it has an identity e, it is a loop (its Cayley table is a Latin square: each row/column is a permutation of e, a, b, c, d).

This loop is not associative

eg. $(ca)d = dd = c$

$c(ad) = cb = e$

c is a left inverse ~~for~~ b ($cb = e$) but not a right inverse ~~for~~ b ($bc = a$).

Theorem If every non-identity element of a group G has order 2, then G is abelian.

Proof (Note: $x^2 = e$ = identity for every $x \in G$.)

Let $x, y \in G$. Then $(xy)^2 = xyxy = e$ so

$$yx = \underbrace{x(x}_{x^2=e}yx\underbrace{y)y}_{y^2=e} = xey = xy. \qquad \square$$

↑ In such groups, $x^{-1} = x$ for all $x \in G$.

Shoe-Sock Theorem

In every group G, with identity 1, for $x, y \in G$ we have $(xy)^{-1} = y^{-1}x^{-1}$.

Proof $(y^{-1}x^{-1})(xy) = y^{-1}1y = 1$ and $(xy)(y^{-1}x^{-1}) = 1$. $\square$

Warning: $(xy)^{-1} \neq x^{-1}y^{-1}$ in general.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

Write the rows of the Cayley table as permutations of $\overset{1}{\overset{\shortparallel}{e}}, \overset{2}{\overset{\shortparallel}{a}}, \overset{3}{\overset{\shortparallel}{b}}, \overset{4}{\overset{\shortparallel}{c}}$:

$\{(), (12)(34), (13)(24), (14)(23)\}$ is a Klein four group as a subgroup of S_4.

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Gives $\{(), (1234), (13)(24), (1432)\}$ as a subgroup of S_4.

Theorem (Cayley Representation Theorem)
Every finite group G is isomorphic to a subgroup of S_n where $n = |G|$.

By the way, every finite group G is also isomorphic to a group of matrices under multiplication.

Theorem If G is a finite group of order n, then every element $g \in G$ has order dividing n. (If $g \in G$ then $|g| \mid n$.).

Eg. S_4 has elements of order $1, 2, 3, 4$. These orders of elements divide $|S_4| = 24$.

S_5 has elements of order $1, 2, 3, 4, 5, 6$ (divisors of $|S_5| = 120$).

Proof In the general case this follows from a later theorem, Lagrange's Theorem. Here let's prove the theorem in the special case that G is abelian. (We have already proved the result for cyclic groups.)

Consider the product of all the group elements $\pi = g_1 g_2 g_3 \cdots g_n$ where $G = \{g_1, g_2, \dots, g_n\}$, $g_1 = 1$.

Note: since G is abelian, π is well-defined; it doesn't depend on what order we list the elements $g_1, \dots, g_n \in G$. Pick $a \in G$. (So $a \in \{g_1, \dots, g_n\}$.) The elements $ag_1, ag_2, \dots, ag_n$ are again all the elements of G so

$$\begin{array}{c|cccc} & g_1 & g_2 & \cdots & g_n \\ \hline a & ag_1 & ag_2 & ag_3 \cdots & ag_n \end{array}$$

$$(ag_1)(ag_2)(ag_3)\cdots(ag_n) = \pi = a^n g_1 g_2 \cdots g_n = a^n \pi$$

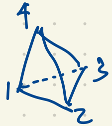
So $a^n = 1$ and $k = |a|$ must divide n. $\square$

Lagrange's Theorem If G is any finite group of order n, and $H \le G$ (i.e. H is a subgroup of G) then $|H| \mid n$.

This generalizes the previous statement: if $g \in G$ then by Lagrange's Theorem, $|g| = |\langle g \rangle| \mid |G|$.

eg. $|A_4| = \frac{1}{2}|S_4| = 12$, $A_4 = \{(), (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$.

The symmetry group of a regular tetrahedron



is isomorphic to S_4.

The rotational symmetry group of the regular tetrahedron (the direct isometry group, consisting of those symmetries that preserve orientation) is isomorphic to A_4.

$A_4 = \{(), (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$.

Subgroups of A_4 have order 1, 2, 3, 4.

Elements of A_4 have order 1, 2, 3.

Divisors of $|A_4| = 12$ are 1, 2, 3, 4, 6, 12.

$\langle (243), (12)(34) \rangle = \{(), (243), (12)(34), (234), (142), (124), \cdots\} = A_4$.

$(243)(12)(34) = (142)$

$\{(), (12)(34), (13)(24), (14)(23)\}$ is the Klein four-group, a subgroup of A_4.

Question: How many subgroups of $\mathbb{Z}$ are there containing 4? (Note: $\mathbb{Z}$ is an additive group.)

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\}$

$2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$

$4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, 12, \cdots\}$

$-4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, 12, \cdots\}$

Answer: There are three subgroups of $\mathbb{Z}$ containing 4, namely $\mathbb{Z}$, $2\mathbb{Z}$, $4\mathbb{Z}$.

$\mathbb{Z}$ has infinitely subgroups: one finite subgroup $\{0\}$ and all the other subgroups are infinite.

There are infinite subgroups of $\mathbb{Z}$ containing 4 but not infinitely many subgroups of $\mathbb{Z}$ containing 4.

Note: For every cyclic group G, all subgroups of G are cyclic; they are generated by powers of the generator of G.

Eg. $G = \langle g \rangle$ where $|g| = \infty$ i.e. $|G| = |\langle g \rangle| = |g| = \infty$.

$= \{ \dots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \dots \}$ with no repeats.

1 is the identity

$g^i g^j = g^{i+j} = g^j g^i$

How many subgroups of $G = \langle g \rangle$ contain g^4? Three: $\overset{\langle g^{-1}\rangle}{\overset{\shortparallel}{\langle g \rangle}}, \overset{\langle g^6, g^{10}\rangle}{\overset{\shortparallel}{\overset{\langle g^{-2}\rangle}{\overset{\shortparallel}{\langle g^2 \rangle}}}}, \overset{\langle g^{-4}\rangle}{\overset{\shortparallel}{\langle g^4 \rangle}}$.

$G = \{ \dots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, g^4, \dots \}$

$\langle g^2 \rangle = \{ \dots, g^{-6}, g^{-4}, g^{-2}, 1, g^2, g^4, g^6, \dots \}$

$\langle g^4 \rangle = \{ \dots, g^{-8}, g^{-4}, 1, g^4, g^8, g^{12}, \dots \}$

$\langle g^6, g^{10} \rangle \leq \langle g^2 \rangle$

$\langle g^2 \rangle \leq \langle g^6, g^{10} \rangle$

since $g^2 = (g^6)^2 (g^{10})^{-1}$

So $\langle g^2 \rangle = \langle g^6, g^{10} \rangle$

$\underbrace{G}_{\text{multiplicative cyclic group}} \cong \underbrace{\mathbb{Z}}_{\text{additive cyclic group}}$

$\phi : \mathbb{Z} \to G$ is an isomorphism

$\phi(i) = g^i$

Theorem If G is a group of even order, then G has an element of order 2 (i.e. at least one element of order 2). Note: G is not necessarily abelian.

Proof Pair up each group element with its inverse giving pairs $\{g, g^{-1}\}$ for $g \in G$.

Note that $g = g^{-1}$ iff g has order 1 or 2. ($g = g^{-1} \iff g^2 = 1 \iff |g|$ divides 2). So G is partitioned into subsets $\{g, g^{-1}\}$ having size 1 or 2. If G has no elements of order 2 then we have partitioned a set G of even cardinality into one subset $\{1\}$ of size 1, and a collection of pairs $\{g, g^{-1}\}$ of size 2, a contradiction. ☐

what we actually showed is that in a group of even order, the number of elements of order 2 is odd. (In a group of odd order, there are no elements of order 2 although we haven't proved this yet except in the abelian case.)

Eg. **Direct Products**: Given groups G, H (say, multiplicative) we form the direct product of G and H as $G\times H = \{(g,h) : g\in G,\ h\in H\}$ (the cartesian product of the sets G and H) which becomes a group under coordinatewise multiplication i.e.

$$(g,h)(g',h') = (gg', hh')$$

and coordinatewise inverses i.e. $(g,h)^{-1} = (g^{-1}, h^{-1})$

and the coordinatewise identity $1\in G\times H$ is $1 = 1_{G\times H} = (1_G, 1_H)$. or $e_{G\times H} = (e_G, e_H)$.

Eg. $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ under addition mod 2

+	0	1
0	0	1
1	1	0

$$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} = \{(x,y) : x,y\in\mathbb{Z}/2\mathbb{Z}\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

$(x,y) + (x',y') = (x+x', y+y')$. The identity $0 = (0,0)$.

This is the Klein four-group since it has 3 elements of order 2.

Note: Many books write $\mathbb{Z}_2$ or Z_2 in place of $\mathbb{Z}/2\mathbb{Z}$

If $|G| = m$ and $|H| = n$ then $|G\times H| = mn$.

If G and H are abelian then so is $G\times H$.

In fact, the converse holds: G and H are both abelian, iff $G\times H$ is abelian.

$G\times H \cong H\times G$

$\phi: G\times H \to H\times G$

$\phi(g,h) = (h,g)$ is an isomorphism.

$G\times H$ has a subgroup $G\times\{1_H\} = \{(g,1_H) : g\in G\} \cong G$

An isomorphism $G\times\{1_H\}\longrightarrow G$ is given by $(g,1_H)\longmapsto g$.

Likewise, $G\times H$ has a subgroup $\{1_G\}\times H \cong H$

$$\begin{tikzcd} (g,1_H) & (1_G,h) \\ G\times\{1_H\} \arrow[u] \arrow[d, phantom, "\cong" rotate=90] & \{1_G\}\times H \arrow[u] \arrow[d, phantom, "\cong" rotate=90] \\ G & H \end{tikzcd}$$

$$\underbrace{(g,1_H)}\underbrace{(1_G,h)} = (g,h) = (1_G,h)(g,1_H)$$

Eg. $\underset{\text{multiplicative group}}{\mathbb{R}^\times = (-\infty,0)\cup(0,\infty)} \cong \underbrace{\mathbb{R}}_{\substack{\text{additive}\\ \text{group}}} \times \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{additive}}$

An isomorphism $\phi:\mathbb{R}^\times\longrightarrow\mathbb{R}\times\mathbb{Z}/2\mathbb{Z}$ is
$$\phi(a) = \begin{cases} (\ln|a|, 0) & \text{if } a>0 \\ (\ln|a|, 1) & \text{if } a<0 \end{cases}$$

It's easy to see that ϕ is one-to-one and onto.

We show that $\phi(ab) = \phi(a)+\phi(b)$ for all $a,b\in\mathbb{R}^\times$.

We argue in four cases. If $a,b>0$ then

$\phi(ab) = (\ln|ab|, 0)$ since $ab>0$

$= (\ln|a|+\ln|b|, 0) = (\ln|a|, 0) + (\ln|b|, 0) = \phi(a)+\phi(b)$

If $a>0>b$ then $ab<0$ so

$\phi(ab) = (\ln|ab|, 1) = (\ln|a|, 0) + (\ln|b|, 1) = \phi(a)+\phi(b)$

Similarly if $a<0<b$.

If $a,b<0$ then $ab>0$ so

$\phi(ab) = (\ln|ab|, 0) = (\ln|a|, 1) + (\ln|b|, 1)$
$= \phi(a)+\phi(b)$

Every cyclic group is abelian.

Not every abelian group is cyclic but every abelian group is a direct product of cyclic groups.

eg. the Klein four-group is a direct product of two groups of order 2 i.e. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

There are five groups of order 8 up to isomorphism:

$$\left.\begin{array}{l} \mathbb{Z}/8\mathbb{Z} \quad \text{(cyclic)} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \{(a,b) : a \in \mathbb{Z}/2\mathbb{Z},\ b \in \mathbb{Z}/4\mathbb{Z}\}. \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(a,b,c) : a,b,c \in \mathbb{Z}/2\mathbb{Z}\} \text{ under addition} \end{array}\right\} \text{three abelian groups of order 8}$$

dihedral group of order 8 $\cong$ symmetry group of square, D_4 (sometimes D_8)

quaternion group of order 8, Q or Q_8

$$Q = \{1, \underset{\substack{\uparrow \\ \text{order 2}}}{-1}, \underbrace{i, -i, j, -j, k, -k}_{\text{order 4}}\}$$

$ij = k,\ ji = -k,\ i^2 = j^2 = k^2 = -1$

$jk = i,\ kj = -i$

$ki = j,\ ik = -j$

For any field F (eg. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$) $GL_n(F) = \{\text{invertible } n \times n \text{ matrices over } F\}$ i.e. having entries in F.

Also $F = \mathbb{F}_3 = \{0,1,2\}$ works with addition mod 3. $2+2 = 1 = 2 \times 2$

$\frac{1}{2} = 2$

In $\mathbb{F}_7 = \{0,1,2,\dots,6\}$, $\frac{1}{5} = 3$.

$\mathbb{F}_p = \{0,1,2,\dots,p-1\}$ is a field whenever p is prime.

$GL_2(\mathbb{F}_3) = \{\text{invertible } 2\times 2 \text{ matrices over } \mathbb{F}_3\}$ is a group of order 48.

$GL_2(\mathbb{R}) = \{\text{invertible } 2\times 2 \text{ matrices over } \mathbb{R}\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad - bc \neq 0\right\}$

$GL_n(F) = \{\text{invertible } n \times n \text{ matrices over } F\}$ = general linear group of degree n over F

also denoted $GL(n,F)$ in the textbook

$SL_n(F)$ is the special linear group of degree n over F ; $SL_n(F) \le GL_n(F)$
or $SL(n,F)$ $\quad SL_n(F) = \{n\times n$ matrices over F having determinant 1$\}$.

If $F = \mathbb{F}_p = \{0,1,2,\dots,p-1\}$ mod p (field of prime order p) then we can count elements in $GL_n(\mathbb{F}_p)$ or $SL_n(\mathbb{F}_p)$. (For 2×2 matrix over $\mathbb{F}_3$, $\underline{33}$ matrices have $\det A = 0$, $\underline{24}$ matrices have $\det A = 1$, $\underline{24}$ . . . . . $\det A = 2$).

$|GL_2(\mathbb{F}_3)| = 48$.

The number of 2×2 matrices over $\mathbb{F}_3 = \{0,1,2\}$ is 81. How many of them are invertible?
We count invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a,b,c,d \in F = \mathbb{F}_3$ with linearly independent columns.
There are $\underline{8} \leftarrow 9-1=8$ choices for the first column $\begin{bmatrix} a \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Having chosen the first column $\begin{bmatrix} a \\ c \end{bmatrix}$, there are $\underline{6} \leftarrow 9-3=6$ choices for the second column $\begin{bmatrix} b \\ d \end{bmatrix}$ which are not a scalar multiple of the first column. So $|GL_2(\mathbb{F}_3)| = 8\times 6 = 48$.
In fact, for $A \in GL_2(F)$, $F = \mathbb{F}_3$, there are 24 choices with determinant 1, and 24 choices with determinant $-1=2$.

$$|GL_n(\mathbb{F}_p)| = \underbrace{(p^n-1)}_{\text{no. of choices of first column}}\underbrace{(p^n-p)}_{\text{no. of choices of second column}}\overbrace{(p^n-p^2)}^{\text{no. of choices of third column}}\cdots\underbrace{(p^n-p^{n-1})}_{\text{no. of choices of last column}}$$

$|GL_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)$

For $A \in GL_n(\mathbb{F}_p)$, $\det A \in \{1,2,\dots,p-1\}$ and there equally many matrices with each possible nonzero determinant in $\{1,2,\dots,p-1\}$ so

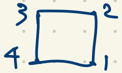
$|SL_n(\mathbb{F}_p)| = \frac{1}{p-1}|GL_n(\mathbb{F}_p)|$. We'll explain later.

For any group G, the center of G is $Z(G) = \{$all elements in G which commute with everything in $G\}$

Zentrum ↑ not $\mathbb{Z}$ Zahlen

$= \{z \in G : zx = xz \text{ for all } x \in G\}$

Eg. if G is the symmetry group of a square (a dihedral group of order 8) then $|Z(G)| = 2$ and $Z(G)$ consists of the identity and the half-turn (180° rotation about the center).



If we represent G using permutations on the vertices 1,2,3,4 then

$$G = \{(\,),\ (1234),\ (13)(24),\ (1432),\ (12)(34),\ (14)(23),\ (13),\ (24)\}$$

then $Z(G) = \langle (13)(24) \rangle = \{(\,),\ (13)(24)\}$.

Alternatively, G can be represented as a subgroup of $GL_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&-1\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&-1\\-1&0\end{bmatrix} \right\}$$

$$Z(G) = \left\langle \begin{bmatrix}-1&0\\0&-1\end{bmatrix} \right\rangle = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix} \right\rangle$$

In general, $Z(G) \le G$ (a subgroup of G).
$Z(G) = G$ iff G is abelian.

For many groups, $Z(G) = \{1\}$ ← identity eg. $Z(S_3) = \{(\,)\}$.

e = identity of G

Theorem If G is a group and $z \in G$, then $Z(G) \le G$ (the center of G is a subgroup of G).

Proof Since $eg = g = ge$ for every $g \in G$, $e \in Z(G)$. If $z, z' \in Z(G)$ then

$$(zz')g = z(z'g) = z(gz') = (zg)z' = (gz)z' = g(zz')$$

so $zz' \in Z(G)$. Also if $z \in Z(G)$ then for every $g \in G$ we have $zg = gz$ so $z^{-1}g = z^{-1}(gz)z^{-1} = z^{-1}(zg)z^{-1} = gz^{-1}$

so $z^{-1} \in Z(G)$. □

Let $S \subseteq G$. The centralizer of S in G is $C_G(S)$ = the set all all elements of G commuting with every element of S, i.e. $C_G(S) = \{g \in G : gs = sg \text{ for all } s \in S\}$.

eg. $C_G(e) = G$, $C_G(G) = Z(G)$. If $z \in Z(G)$ then $C_G(z) = G$.

In S_4, $C_{S_4}((12)) = \{(), (34), (12), (12)(34)\}$

In general, $C_G(S) \leq G$ (the centralizer of a subset of G is always a subgroup of G).
The proof of this is virtually identical to the proof above; just quantify over $g \in S$ rather than $g \in G$.

If $G = GL_n(F)$ = invertible $n \times n$ matrices over F, then $Z(G) = \{\lambda I : \lambda \neq 0 \text{ in } F\}$

$\uparrow$ $I = I_n$ = $n \times n$ identity matrix.

$$\left.\begin{aligned}\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} &= \begin{bmatrix}1&1\\0&2\end{bmatrix}\\ \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix} &= \begin{bmatrix}1&2\\0&2\end{bmatrix}\end{aligned}\right\} \text{ so } \begin{bmatrix}1&0\\0&2\end{bmatrix} \notin Z(GL_2(\mathbb{Q}))$$

Let $E_{ij}(a) = \begin{array}{cc} & j \\ i & \begin{bmatrix}1 & & a\\ & \ddots & \\ & & 1\end{bmatrix}\end{array}$ for $i \neq j$. (This is the elementary matrix obtained from the identity matrix by adding an "a" in the (i,j) position.)

If $A = [a_{ij} : 1 \leq i, j \leq n] \in Z(GL_n(F))$ then $A E_{ij}(1) = E_{ij}(1) A$ so $a_{ij} = 0$. So A is diagonal.
Continue using other elementary matrices to show $A = \lambda I$.

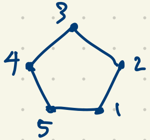
$G = GL_n(F)$ is generated by elementary matrices so $A \in Z(G)$ iff A commutes with all elementary matrices.

$Z(G)$ might be trivial e.g. $Z(S_3) = \{()\}$.

Another construction of subgroups: Suppose $G \leq S_n$. So G permutes $[n] = \{1, 2, \dots, n\}$.

The <u>stabilizer</u> of a point $x \in [n]$ is $\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\} \leq G$.

Eg.



The symmetry group of a regular pentagon is a group G which is dihedral of order 10 (sometimes denoted D_5 or D_{10}).

$$G = \{\underbrace{(), (12345), (13524), (14253), (15432)}_{5\text{ rotations}}, \underbrace{(12)(35), (13)(45), (14)(23), (15)(24), (25)(34)}_{5\text{ reflections}}\}$$

$G \leq S_5$ permuting $[5] = \{1, 2, 3, 4, 5\}$, the five vertices.

$\mathrm{Stab}_G(3) = \{(), (15)(24)\}$.

$()(x) = x$

If $g, h \in \mathrm{Stab}_G(x)$ then

$$(gh)(x) = g(h(x)) = g(x) = x$$

If $g \in \mathrm{Stab}_G(x)$ then $g(x) = x$ so

$$x = g^{-1}(g(x)) = g^{-1}(x) \quad \text{so} \quad g^{-1} \in \mathrm{Stab}_G(x).$$