Math 3500

Algebra I: Group Theory

Book 2

Similar to HW#2: How many elements of each order does S_4 have?

1 element of order 1 : $()$ = identity,

9 elements of order 2 : (12), (13), (14), (23), (24), (34), $\leftarrow$ $\binom{4}{2} = 6$ transpositions

$(12)(34)$, $(13)(24)$, $(14)(23)$ $\leftarrow$ $\frac{1}{2!}\binom{4}{2}\binom{2}{2} = \frac{6}{2} = 3$

8 elements of order 3: (123), (124), (132), (134), (142), (143), (234), (243)

6 elements of order 4: (1234), (1243), (1324), (1342), (1423), (1432)

$24 = 4! = |S_4|$

$|S_n| = n! = 1 \times 2 \times 3 \times \cdots \times n$

In S_n the number of n-cycles is $(n-1)!$

The binomial coefficient $\binom{n}{k}$ ("n choose k") is the number of ways to choose a subset of size k from a set of size n.

By the way, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\binom{n}{k} = k^{th}$ entry in n^{th} row of Pascal's Triangle

Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\binom{4}{2}$ = number of 2-subsets in $[4] = \{1,2,3,4\}$
$= 6$

Pascal's Triangle

	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
n=0	1							
n=1	1	1						
n=2	1	2	1					
n=3	1	3	3	1				
n=4	1	4	6	4	1			
n=5	1	5	10	10	5	1		
n=6	1	6	15	20	15	6	1	
n=7	1	7	21	35	35	21	7	1

A ~~transposition~~ is a 2-cycle $(i\ j) \in S_n$, $i \neq j$ in $[n] = \{1, 2, \cdots, n\}$.

Products of disjoint transpositions e.g. $(1\ 3)(2\ 5)(6\ 8) \in S_8$ are elements of order 2.

How many elements of order 2 are there in S_7?

Transpositions: $(12), (13), (14), \cdots, (67)$ i.e. $(i\ j)$ where $i \neq j$ in $[7] = \{1, 2, \cdots, 7\}$

$\binom{7}{2} = 21$ transpositions

Products of two disjoint transpositions e.g. $(2\,6)(3\,4) = (3\,4)(2\,6)$

Number of these is $\frac{1}{2}\underbrace{\binom{7}{2}}_{21}\underbrace{\binom{5}{2}}_{10} = 105$.

Products of three disjoint transpositions eg. $(1\,5)(27)(36) = (15)(36)(27) = (27)(15)(36) = \cdots$

Number of these is $\underbrace{\frac{1}{3!}}_{\frac{1}{6}}\underbrace{\binom{7}{2}}_{21}\underbrace{\binom{5}{2}}_{10}\underbrace{\binom{3}{2}}_{3} = \frac{21 \times 10 \times \cancel{3}}{\cancel{6}\ 2} = 105$

Number of 3-cycles in S_7 eg. $(2\,7\,4)$: $2\binom{7}{3} = 2 \times 35 = 70$

Number of products of two disjoint 3-cycles: eg. $(274)(356) = (356)(274)$

$\frac{1}{\cancel{2}} \cdot 70 \cdot \binom{4}{3} \cdot \cancel{2} = 70 \cdot 4 = 280$

$280 + 70 = 350$ elements of order 3 in S_7

Elements of order 12 in S_7 eg. $(1\,4\,2)(3\,7\,5\,6)$

$70 \cdot 3! = 70 \cdot 6 = 420$ elements of order 12 in S_7.