Algebra I

Group Theory

Book 1

A <u>group</u> is a set G with a binary operation $*$ which has an identity element; the operation is associative; and every element has an inverse.

Eg. $\mathbb{R}$ = set of real numbers under addition '+'. Its identity element is 0.

$$\left.\begin{array}{l} 0+x=x \\ (x+y)+z = x+(y+z) \\ x+(-x) = 0 = (-x)+x \end{array}\right\} \quad \text{for all } x,y,z\in\mathbb{R}$$

$(\mathbb{R}, +)$ is a group.

$(\mathbb{R}, \times)$ (real numbers under multiplication is almost but not quite a group. (0 does not have an inverse). 1 is the identity.

$\mathbb{R}^\times$ = {all nonzero real numbers} $= \{a\in\mathbb{R} : a\neq 0\}$ is a group under multiplication.

$$\left.\begin{array}{l} 1a=a \\ (ab)c = a(bc) \\ a\cdot a^{-1} = a^{-1}\cdot a = 1 \qquad a^{-1}=\frac{1}{a} \end{array}\right. \quad \text{for all } a,b,c\in\mathbb{R}^\times.$$

$(\mathbb{R}^\times, \times)$ is a group.

$\mathbb{R}$ with the operation $x*y = x+y+7$. This is a group $(\mathbb{R}, *)$. For all $x,y,z\in\mathbb{R}$,

$$(x*y)*z = (x+y+7)+z+7 = x+y+z+14 = x+(y+z+7)+7 = x*(y*z)$$

so $(\mathbb{R}, *)$ is associative. Note that $-7\in\mathbb{R}$ is an identity element since

$$\left.\begin{array}{l} -7*x = (-7)+x+7 = x \\ \text{and} \quad x*(-7) = x+(-7)+7 = x \end{array}\right\} \quad \text{for all } x\in\mathbb{R}.$$

So $-7\in\mathbb{R}$ is an identity element for '$*$'.

$$\left.\begin{array}{l} (-x-14)*x = (-x-14)+x+7 = -7 \\ x*(-x-14) = x+(-x-14)+7 = -7 \end{array}\right\} \quad \text{for all } x\in\mathbb{R}.$$

So $-x-14$ is an inverse element for x.

$$(x*y)*z = x*(y*z)$$
$$\Leftrightarrow (x+y+7)+z+7 = x+(y+z+7)+7$$
$$\Leftrightarrow x+y+z+14 = x+y+z+14$$

so $(\mathbb{R}, *)$ is associative.

$$7 = 3$$
$$\Rightarrow 7-5 = 3-5$$
$$\Rightarrow 2 = -2$$
$$\Rightarrow (2)^2 = (-2)^2$$
$$\Rightarrow 4 = 4$$

$$\begin{aligned}(x*y)*z &= (x+y+7)+z+7\\ &= x+y+z+14\\ &= x+(y+z+7)+7\\ &= x*(y*z)\end{aligned}$$

$(\mathbb{Q}, +)$ is a group. $\quad \mathbb{Q} = \{\text{rational numbers}\}$

$-\frac{5}{3} \in \mathbb{Q}$

$\frac{172}{100} = 1.72 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

$(\mathbb{Q}^\times, \times)$ is a group.

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\} = \{\text{all nonzero rational numbers}\}$

$(\mathbb{N}, +)$ is not a group

$\mathbb{N} = \{1, 2, 3, 4, \cdots\} = \mathbb{Z}^{>0}$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \cdots\} = \mathbb{Z}^{\geq 0}$

$\mathbb{Z} = \{\text{integers}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$(\mathbb{Z}, +)$ is a group.

$(\mathbb{Z}, +) \underset{\text{subgroup}}{\leq} (\mathbb{Q}, +) \underset{\text{subgroup}}{\leq} (\mathbb{R}, +) \leq (\mathbb{C}, +)$; but $(\mathbb{R}^\times, \times)$ is not a subgroup $(\mathbb{R}, +)$ $\not\leq$ (although $\mathbb{R}^\times \underset{\text{subset}}{\subseteq} \mathbb{R}$)

In $\mathbb{R}^\times$, $2 \times 3 = 6$ but in $(\mathbb{R}, +)$, $2+3=5$

$GL_n(\mathbb{R}) = \{\text{invertible } n\times n \text{ matrices with real entries}\}$ is the <u>general</u> <u>linear</u> <u>group</u>

$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$GL_n(\mathbb{R})$ is a multiplicative group with identity $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$GL_n(\mathbb{R})$ is <u>not</u> commutative for $n \geq 2$.

$GL_1(\mathbb{R})$ is commutative.

$(G, *)$ is <u>Abelian</u> (abelian) if $x * y = y * x$ for all $x, y \in G$.

$GL_n(\mathbb{R})$ is abelian for $n=1$; nonabelian for $n \geq 2$. $\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 5 & 35 \end{bmatrix}$ whereas $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 38 \end{bmatrix}$.

$GL_1(\mathbb{R}) \cong \mathbb{R}^\times$ (these are isomorphic groups i.e. essentially the same group. Since $\mathbb{R}^\times$ is abelian, so is $GL_1(\mathbb{R})$.)

Function composition is associative: $(f\circ g)\circ h = f\circ(g\circ h)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & Z \arrow[r, "f"] & W \end{tikzcd}$$

If $x \in X$ then $h(x) \in Y$, $g(h(x)) \in Z$, $\underbrace{f(g(h(x)))}_{(f\circ g\circ h)(x)} \in W$.

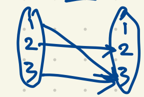
$f\circ g \neq g\circ f$

Because matrix multiplication is expressing the composition of linear transformations, it is associative but not necessarily commutative.

If X is any set, the bijections $X \xrightarrow{f} X$ (i.e. f one-to-one and onto) form a group under composition. This is the <u>symmetric group</u>

$$G = \mathrm{Sym}\,X = \{\text{bijections } X \to X\} = \{\text{permutations of } X\}.$$

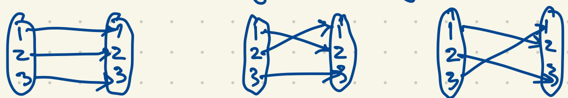
Not a bijection (neither one-to-one nor onto)



eg. $X = [3] = \{1,2,3\}$. (Notation: $[n] = \{1,2,3,\dots,n\}$.)

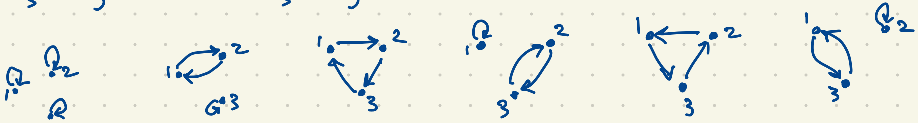
There are exactly $3! = 6$ bijections $[3] \to [3]$.

$n! = 1\times2\times3\times\cdots\times n$ (n factorial) is the number of permutations of $[n]$.



x	$f(x)$
1	1
2	2
3	3

x	$f(x)$
1	2
2	1
3	3



( ) (1 2) (1 2 3) (2 3) (1 3 2) (1 3)

$|S_3| = 6$. S_3 is a nonabelian group of order 6.

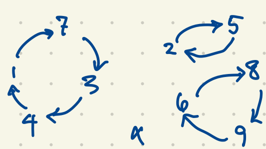
S_3 is the smallest nonabelian group.

In S_3, $(12)(13) = (132)$
$(13)(12) = (123)$

cycle notation for $\mathrm{Sym}\,[3] = S_3$
$= \{(\,),\ (12),\ (13),\ (23),\ (123),\ (132)\}$

(12)(3)

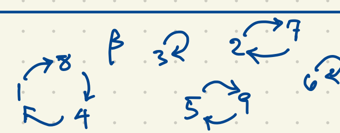
Eg. $n = 9$



α

n	$\alpha(n)$	$\beta(n)$	$\alpha\beta(n)$
1	7	8	9
2	5	7	3
3	4	3	4
4	1	1	7
5	2	9	6
6	8	6	8
7	3	2	5
8	9	4	1
9	6	5	2

β



$\beta = (7,2)(4,1,8)(3)(6)(5,9) = (1\ 8\ 4)(2\ 7)(5\ 9)$

$(7,2) = (2,7)$

$(4,1,8) = (1,8,4) = (8,4,1)$

$(3) = (\,)$

$\alpha = (1,7,3,4)(2,5)(6,8,9)$

$\alpha\beta = \alpha\circ\beta = (1,9,2,3,4,7,5,6,8) = (1734)(25)(689)(184)(27)(59)$

$\beta\alpha = \beta\circ\alpha = (129648573) = (184)(27)(59)(1734)(25)(689)$

If α, β are permutations then $\alpha\beta \neq \beta\alpha$ in general but they have the same cycle structure.

The order of a group G is $|G|$, the number of elements in the group. (finite or infinite)

$|S_n| = n!$

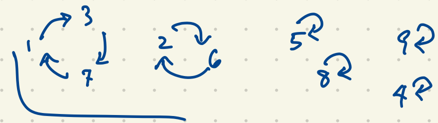
$|GL_n(\mathbb{R})| = \infty$

S_n is nonabelian for $n \geq 3$.

$S_2 = \{(),\ (12)\}$ is abelian.

In S_n, disjoint cycles always commute, e.g. in S_9, $(137)(26) = (26)(137)$

If two permutations commute, must they have disjoint cycles?



$\alpha = (135)(246)$

$\beta = (12)(34)(56)$

Note: The two 3-cycles in α intersect with the three 2-cycles in β.

$\alpha\beta = (135)(246)(12)(34)(56) = (145236)$

$\beta\alpha = (12)(34)(56)(135)(246) = (145236)$

S_n acts on $[n] = \{1, 2, \dots, n\}$ (the n points that we are permuting)

Do not confuse S_n with $[n]$. THIS IS NOT THAT. $|S_n| = n!$, $|[n]| = n$.

Typically, groups act on things (generically called points).

Typically, groups describe symmetries of things.

A cube has 48 symmetries forming a group G of order 48. $|G| = 48$.

24 of these are direct symmetries preserving orientation: these are rotations.

24 of these are virtual symmetries which cannot be obtained by physical motion.

number of edges

$12 \times 4 = 48$

number of symmetries fixing each edge

number of vertices

$8 \times 6 = 48$

number of symmetries fixing each vertex

how many symmetries map each face to each of the other faces

$6 \times 8 = 48$

number of faces

In a group G with identity e, an element $g \in G$ has <u>order</u> n if $\underbrace{g^n}_{\underbrace{g * g * \cdots * g}_{n \geq 1}} = e$
but no smaller power of equals e.
If G is the symmetry group of a cube, every reflection has order 2.
Also a 180° rotation about any axis has order 2.
A 120° rotation of the cube about an axis joining two opposite (antipodal) vertices has order 3.
The cube has axes of symmetry joining centers of opposite faces, and a 90° rotation around such an axis has order 4.
In any group, the identity has order 1.

S_3 has 1 element of order 1, i.e. ()
3 elements of order 2, i.e. (12), (13), (23)
2 elements of order 3, i.e. (132), (123)

$|S_3| = 6$

The order of an n-cycle. If $\alpha = (1,2,3,\dots,n)$ then $\alpha^n = ()$ but $\alpha^k \neq ()$ for $k = 1,2,\dots,n-1$.

S_4 has 1 elements of order 1, i.e. ()
9 . . . . . . 2, i.e. (12), ..., (13)(24), ... (six $= \binom{4}{2}$ 2-cycles $(i\,j)$; three permutations $(i\,j)(k\,l)$ having the same cycle structure as (13)(24))
8 . . . . . . 3, i.e. (123), ... (eight 3-cycles $(i\,j\,k)$, the same " " " (123))
6 . . . . . . 4, six 4-cycles e.g. (1234)

$|S_4| = 24$

$$\binom{m}{n} = \binom{m}{m-n}$$

$\binom{m}{n}$ = number of n-subsets of an m-set, e.g. $\binom{4}{2} = 6$: a 4-set (set with 4 elements, e.g. $[4] = \{1,2,3,4\}$) has six subsets of size 2 : $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

$$= \frac{m!}{n!\,(m-n)!} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n(n-1)(n-2)\cdots 1}$$

so $\binom{4}{2} = \frac{4\cdot 3}{2\cdot 1} = 6$. $\binom{4}{3}\cdot 2 = 4\cdot 2 = 8$ $\binom{4}{3} = \frac{4\cdot\cancel{3}\cdot\cancel{2}}{\cancel{3}\cdot\cancel{2}\cdot 1} = 4$

$S_5 = \{$ permutations of $[5] = \{1,2,3,4,5\}\}$ is a group of order $|S_5| = 5! = 120$.

$(1\ 2)(1\ 3) = (1\ 3\ 2)$

How many elements of each order does S_5 have?

$\underline{1}$ element of order 1 : $(\ )$

$\underline{25}$ elements of order 2 : $(i\ j)$ $\binom{5}{2} = 10$ cycles of length 2

$(i\ j)(k\ l)$ $5 \times 3 = 15$ elements which are a product of two disjoint 2-cycles

or: $\underbrace{10}_{\substack{\text{how many}\\ \text{choices of}\\ \text{2-cycle } (i\ j)}} \times \underbrace{3}_{\substack{\text{how many}\\ \text{2-cycles } (k\ l)\\ \text{disjoint from } (i\ j)}} \div 2 = 15$

since $(i\ j)(k\ l) = (k\ l)(i\ j)$

A 2-cycle $(i\ j)$ (i.e. cycle of length 2) is a <u>transposition</u>.

$\underline{20}$ elements of order 3 : 3-cycles $(i\ j\ k)$ $\binom{5}{3} \times 2 = 10 \times 2 = 20$

$\underline{30}$ elements of order 4 : 4-cycles $(i\ j\ k\ l)$ e.g. $(1\ 2\ 3\ 4)$, $(1\ 3\ 4\ 2)$, $(2\ 5\ 3\ 4)$, ...

$\binom{5}{4} \times 3! = 5 \times 6 = 30$

how many ways to choose i, j, k, l

$\underline{24}$ elements of order 5 : 5-cycles $(1\ \underbrace{*\ *\ *\ *}_{2,3,4,5})$

$\underline{20}$ elements of order 6 : $(i\ j\ k)(l\ m)$

$(1\ 2\ 3)(4\ 5) \in S_5$ has order 6

$(1\ 2\ 3\ 4)(5\ 6) \in S_6$ has order 4

$120 = |S_5|$

If $\alpha \in S_n$ is written as a product of disjoint cycles, then its order is the least common multiple of the lengths of its cycles.

$(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)$ has order 15

$(1\ 2\ 3)(4\ 5\ 6\ 7\ 8\ 9)$ 6

In $\mathbb{R}^\times = \{$ nonzero real numbers $\}$ under multiplication,

1 has order 1;

-1 " " 2; $(-1 \cdot -1 = 1)$

every other element of $\mathbb{R}^\times$ has infinite order.

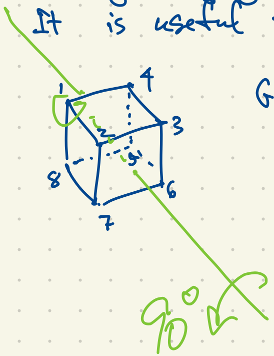
If $a \in \mathbb{R}^\times$, $\quad \operatorname{ord}(a) = \begin{cases} 1, & \text{if } a = 1; \\ 2, & \text{if } a = -1; \\ \infty, & \text{otherwise.} \end{cases}$

We also write the order of $a \in G$ as $|a|$ e.g.

$|(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)| = 15$

$\operatorname{ord}((1\ 2\ 3)(4\ 5\ 6\ 7\ 8)) = 15$

The symmetry group of a cube is a group G of order 48 i.e. $|G| = 48$.
It is useful to think of G as a subgroup of S_8:



$$G = \{ \underbrace{(\,)}_{\text{identity}},\ \underbrace{(1234)(5876)}_{\substack{90^\circ \text{ rotation about} \\ \text{the vertical axis} \\ \text{of symmetry}}},\ \underbrace{(1854)(2763)}_{\substack{90^\circ \text{ rotation} \\ \text{about green} \\ \text{axis of symmetry}}},\ \underbrace{(18)(27)(36)(45)}_{\substack{\text{reflection in} \\ \text{horizontal plane} \\ \text{of symmetry}}},\ (173)(486),\ \dots \}$$

$(1854)(2763)(1234)(5876) = (173)(2)(486)(5) = (173)(486)$ is a 120° rotation about the axis joining the pair of antipodal vertices 2,5

If G is any group and $g_1, \dots, g_k \in G$ then $\langle g_1, g_2, \dots, g_k \rangle$ = the subgroup of G generated by $g_1, \dots, g_k$ i.e. the smallest subgroup of G containing $g_1, \dots, g_k$.

The letter S has a rotational symmetry about its centre (rotate 180° about $\to$ S. The symmetry group in this case is $\{I, R\}$ where R is the 180° rotation, $R^2 = I$. Both symmetries of S preserve orientation. S ≠ S

U ≠ U

U has symmetry group of order 2 $\{I, T\}$ where T is a reflection in the vertical axis of symmetry, $T^2 = I$

Reflections reverse orientation; rotations preserve orientation.

Y has symmetry group of order 2

Y has symmetry group of order 1.

 has symmetry group of order 6. (3 rotational symmetries, 3 reflective symmetries).

For any object $X \subset \mathbb{R}^n$, either all symmetries of X preserve orientation or exactly half of the symmetries preserve orientation (so the other half reverse orientation).

reflection about vertical axis of symmetry

axis Y

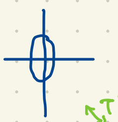
The symmetry group of Y is $\{\underbrace{I, R}_{\substack{\text{(counter-clockwise } 120^\circ \\ \text{rotation about center)}}}, R^2, T, TR, TR^2\} = \langle T, R\rangle$

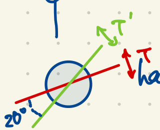
Y

$RT = TR^2$ so the group is nonabelian.

$$\text{E} \neq \text{Ǝ}$$

The figure EƎ as a symmetry group of order 4 $\{I, R, T, RT\}$ where I = identity, $R = 180^\circ$ rotation about the center, T = reflection in horizontal axis of symmetry, $RT = TR$ = reflection in the vertical axis of symmetry. This group is abelian.

 O has the same symmetry group as EƎ (abelian of order 4).



has infinitely many symmetries. The symmetry group is infinite nonabelian.

$$\underbrace{TT'}_{\substack{40^\circ \text{ rotation} \\ \text{clockwise} \\ \text{(i.e. } 320^\circ \text{ counterclockwise} \\ \text{rotation about center)}}} \neq \underbrace{T'T}_{\substack{40^\circ \text{ rotation} \\ \text{counterclockwise} \\ \text{about center}}}$$