Algebra I

Group Theory

Book 1

A <u>group</u> is a set G with a binary operation $*$ which has an identity element; the operation is associative; and every element has an inverse.

Eg. $\mathbb{R}$ = set of real numbers under addition '$+$'. Its identity element is 0.

$$\left.\begin{aligned} &0 + x = x \\ &(x+y)+z = x+(y+z) \\ &x+(-x) = 0 = (-x)+x \end{aligned}\right\} \quad \text{for all } x,y,z \in \mathbb{R}$$

$(\mathbb{R}, +)$ is a group.

$(\mathbb{R}, \times)$ (real numbers under multiplication is almost but not quite a group. (0 does not have an inverse). 1 is the identity.

$\mathbb{R}^\times$ = {all nonzero real numbers} $= \{a \in \mathbb{R} : a \neq 0\}$ is a group under multiplication.

$$\begin{aligned} &1a = a \\ &(ab)c = a(bc) \\ &a \cdot a^{-1} = a^{-1} \cdot a = 1 \qquad a^{-1} = \tfrac{1}{a} \end{aligned} \qquad \text{for all } a,b,c \in \mathbb{R}^\times.$$

$(\mathbb{R}^\times, \times)$ is a group.

$\mathbb{R}$ with the operation $x * y = x + y + 7$. This is a group $(\mathbb{R}, *)$. For all $x,y,z \in \mathbb{R}$,

$$(x*y)*z = (x+y+7)+z+7 = x+y+z+14 = x+(y+z+7)+7 = x*(y*z)$$

so $(\mathbb{R}, *)$ is associative. Note that $-7 \in \mathbb{R}$ is an identity element since

$$\left.\begin{aligned} &-7 * x = (-7)+x+7 = x \\ \text{and}\quad &x*(-7) = x+(-7)+7 = x \end{aligned}\right\} \quad \text{for all } x \in \mathbb{R}. \qquad \text{So } -7 \in \mathbb{R} \text{ is an identity element for '}*\text{'}.$$

$$\left.\begin{aligned} &(-x-14)*x = (-x-14)+x+7 = -7 \\ &x*(-x-14) = x+(-x-14)+7 = -7 \end{aligned}\right\} \quad \text{for all } x \in \mathbb{R}. \qquad \text{So } -x-14 \text{ is an inverse element for } x.$$

$$(x*y)*z = x*(y*z)$$
$$\Leftrightarrow (x+y+7)+z+7 = x+(y+z+7)+7$$
$$\Leftrightarrow x+y+z+14 = x+y+z+14$$

so $(\mathbb{R}, *)$ is associative.

$$7 = 3$$
$$\Rightarrow 7-5 = 3-5$$
$$\Rightarrow 2 = -2$$
$$\Rightarrow (2)^2 = (-2)^2$$
$$\Rightarrow 4 = 4$$

$$(x*y)*z = (x+y+7)+z+7$$
$$= x+y+z+14$$
$$= x+(y+z+7)+7$$
$$= x*(y*z)$$

$(\mathbb{Q}, +)$ is a group. $\quad \mathbb{Q} = \{\text{rational numbers}\}$ $\quad -\frac{5}{3} \in \mathbb{Q}$

$\frac{172}{100} = 1.72 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

$(\mathbb{Q}^\times, \times)$ is a group.

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\} = \{\text{all nonzero rational numbers}\}$

$(\mathbb{N}, +)$ is not a group

$\mathbb{N} = \{1, 2, 3, 4, \cdots\} = \mathbb{Z}^{>0}$

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \cdots\} = \mathbb{Z}^{\geq 0}$

$\mathbb{Z} = \{\text{integers}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$(\mathbb{Z}, +)$ is a group.

$(\mathbb{Z}, +) \underset{\text{subgroup}}{\leq} (\mathbb{Q}, +) \underset{\text{subgroup}}{\leq} (\mathbb{R}, +) \leq (\mathbb{C}, +)$; but $(\mathbb{R}^\times, \times)$ is not a subgroup $(\mathbb{R}, +)$ $\quad \not\leq$ (although $\mathbb{R}^\times \underset{\text{subset}}{\subseteq} \mathbb{R}$)

In $\mathbb{R}^\times$, $2 \times 3 = 6$ but in $(\mathbb{R}, +)$, $2+3 = 5$

$GL_n(\mathbb{R}) = \{\text{invertible } n\times n \text{ matrices with real entries}\}$ is the <u>general linear group</u>

$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$GL_n(\mathbb{R})$ is a multiplicative group with identity $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$GL_n(\mathbb{R})$ is <u>not</u> commutative for $n \geq 2$.

$GL_1(\mathbb{R})$ is commutative.

$(G, *)$ is <u>Abelian</u> (abelian) if $x * y = y * x$ for all $x, y \in G$.

$GL_n(\mathbb{R})$ is abelian for $n=1$; nonabelian for $n \geq 2$. $\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 5 & 35 \end{bmatrix}$ whereas $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 38 \end{bmatrix}$.

$GL_1(\mathbb{R}) \cong \mathbb{R}^\times$ (these are isomorphic groups i.e. essentially the same group. Since $\mathbb{R}^*$ is abelian, so is $GL_1(\mathbb{R})$.)

Function composition is associative: $(f\circ g)\circ h = f\circ(g\circ h)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & Z \arrow[r, "f"] & W \end{tikzcd}$$

If $x \in X$ then $h(x) \in Y$, $g(h(x)) \in Z$, $\underbrace{f(g(h(x)))}_{(f\circ g\circ h)(x)} \in W$.

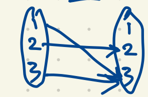
$f\circ g \neq g\circ f$

Because matrix multiplication is expressing the composition of linear transformations, it is associative but not necessarily commutative.

If X is any set, the bijections $X \xrightarrow{f} X$ (i.e. f one-to-one and onto) form a group under composition. This is the symmetric group

$$G = \mathrm{Sym}\, X = \{\text{bijections } X \to X\} = \{\text{permutations of } X\}.$$

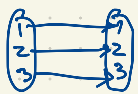
Not a bijection (neither one-to-one nor onto)

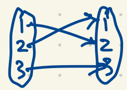


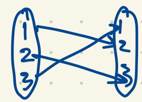
eg. $X = [3] = \{1,2,3\}$. (Notation: $[n] = \{1,2,3,\dots,n\}$.)

There are exactly $3! = 6$ bijections $[3] \to [3]$.

$n! = 1 \times 2 \times 3 \times \cdots \times n$ (n factorial) is the number of permutations of $[n]$.







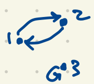
x	$f(x)$
1	1
2	2
3	3

x	$f(x)$
1	2
2	1
3	3

$|S_3| = 6$. S_3 is a nonabelian group of order 6. S_3 is the smallest nonabelian group.

In S_3, $(12)(13) = (132)$
$(13)(12) = (123)$


$(\ )$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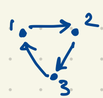
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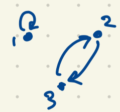
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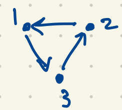
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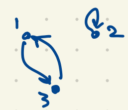
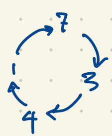
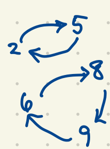

(13)

cycle notation for $\mathrm{Sym}[3] = S_3$
$= \{(\ ), (12), (13), (23), (123), (132)\}$ ← $(12)(3)$

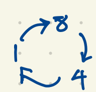
Eg. $n = 9$



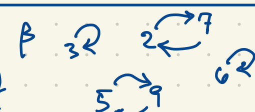
α



n	$\alpha(n)$	$\beta(n)$	$\alpha\beta(n)$
1	7	8	9
2	5	7	3
3	4	3	4
4	1	1	7
5	2	9	6
6	8	6	8
7	3	2	5
8	9	4	1
9	6	5	2



β



$\beta = (7,2)(4,1,8)(3)(6)(5,9) = (1\,8\,4)(2\,7)(5\,9)$

$(7,2) = (2,7)$

$(4,1,8) = (1,8,4) = (8,4,1)$

$(3) = (\ )$

$\alpha = (1,7,3,4)(2,5)(6,8,9)$

$\alpha\beta = \alpha \circ \beta = (1,9,2,3,4,7,5,6,8) = (1734)(25)(689)(184)(27)(59)$

$\beta\alpha = \beta \circ \alpha = (129648573) = (184)(27)(59)(1734)(25)(689)$

If α, β are permutations then $\alpha\beta \neq \beta\alpha$ in general but they have the same cycle structure.

The order of a group G is $|G|$, the number of elements in the group. (finite or infinite)

$|S_n| = n!$

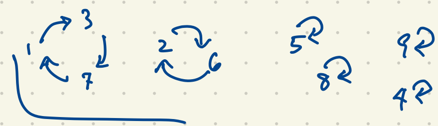
$|GL_n(\mathbb{R})| = \infty$

S_n is nonabelian for $n \geq 3$.

$S_2 = \{(), (12)\}$ is abelian.

In S_n, disjoint cycles always commute, e.g. in S_9, $(137)(26) = (26)(137)$

If two permutations commute, must they have disjoint cycles?



$\alpha = (135)(246)$

$\beta = (12)(34)(56)$

Note: The two 3-cycles in α intersect with the three 2-cycles in β.

$\alpha\beta = (135)(246)(12)(34)(56) = (145236)$

$\beta\alpha = (12)(34)(56)(135)(246) = (145236)$

S_n acts on $[n] = \{1, 2, \ldots, n\}$ (the n points that we are permuting)

Do not confuse S_n with $[n]$. THIS IS NOT THAT. $|S_n| = n!$, $|[n]| = n$.

Typically, groups act on things (generically called points).

Typically, groups describe symmetries of things.

A cube has 48 symmetries forming a group G of order 48. $|G| = 48$.

24 of these are direct symmetries preserving orientation: these are rotations.

24 of these are virtual symmetries which cannot be obtained by physical motion.

number of edges

$12 \times 4 = 48$

number of symmetries fixing each edge

number of vertices

$8 \times 6 = 48$

number of symmetries fixing each vertex

how many symmetries map each face to each of the other faces

$6 \times 8 = 48$

number of faces

In a group G with identity e, an element $g \in G$ has <u>order</u> n if $\underbrace{g^n}_{\underbrace{g * g * \cdots * g}_{n \geq 1}} = e$
but no smaller power of g equals e.

If G is the symmetry group of a cube, every reflection has order 2.

Also a $180°$ rotation about any axis has order 2.

A $120°$ rotation of the cube about an axis joining two opposite (antipodal) vertices has order 3.

The cube has axes of symmetry joining centers of opposite faces, and a $90°$ rotation around such an axis has order 4.

In any group, the identity has order 1.

S_3 has
- 1 element of order 1, i.e. $()$
- 3 elements of order 2, i.e. (12), (13), (23)
- 2 elements of order 3, i.e. (132), (123)

$|S_3| = 6$

The order of an n-cycle. If $\alpha = (1, 2, 3, \cdots, n)$ then $\alpha^n = ()$ but $\alpha^k \neq ()$ for $k = 1, 2, \cdots, n-1$.

S_4 has
- 1 elements of order 1, i.e. $()$
- 9 $\cdots\cdots$ 2, i.e. $(12), \cdots, (13)(24), \cdots$ (six $= \binom{4}{2}$ 2-cycles $(i\,j)$; three permutations $(i\,j)(k\,l)$ having the same cycle structure as $(13)(24)$)
- 8 $\cdots\cdots$ 3, i.e. $(123), \cdots$ (eight 3-cycles $(i\,j\,k)$, the same " " " (123))
- 6 $\cdots\cdots$ 4, six 4-cycles e.g. (1234)

$|S_4| = 24$

$$\binom{m}{n} = \binom{m}{m-n}$$

$\binom{m}{n}$ = number of n-subsets of an m-set, e.g. $\binom{4}{2} = 6$: a 4-set (set with 4 elements, e.g. $[4] = \{1,2,3,4\}$) has six subsets of size 2 : $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$

$$= \frac{m!}{n!\,(m-n)!} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n(n-1)(n-2)\cdots 1}$$

so $\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$.

$\binom{4}{3} \cdot 2 = 4 \cdot 2 = 8$

$\binom{4}{3} = \frac{4 \cdot \cancel{3} \cdot \cancel{2}}{\cancel{3} \cdot \cancel{2} \cdot 1} = 4$

$S_5 = \{\text{permutations of } [5] = \{1,2,3,4,5\}\}$ is a group of order $|S_5| = 5! = 120$.

$(1\ 2)(1 3) = (1 3\ 2)$

How many elements of each order does S_5 have?

$\underline{1}$ element of order 1: $(\ )$

$\underline{25}$ elements of order 2: $(i\ j)$ $\binom{5}{2} = 10$ cycles of length 2

$(i\ j)(k\ l)$ $5 \times 3 = 15$ elements which are a product of two disjoint 2-cycles

or: $\underbrace{10}_{\substack{\text{how many}\\\text{choices of}\\\text{2-cycle } (i\ j)}} \times \underbrace{3}_{\substack{\text{how many}\\\text{2-cycles } (k\ l)\\\text{disjoint from } (i\ j)}} \div 2 = 15$

$\div 2$ since $(i\ j)(k\ l) = (k\ l)(i\ j)$

A 2-cycle $(i\ j)$ (i.e. cycle of length 2) is a __transposition__.

$\underline{20}$ elements of order 3: 3-cycles $(i\ j\ k)$ $\binom{5}{3} \times 2 = 10 \times 2 = 20$

$\underline{30}$ elements of order 4: 4-cycles $(i\ j\ k\ l)$ e.g. $(1\,2\,3\,4)$, $(1\,3\,4\,2)$, $(2\,5\,3\,4)$, ...

$\underline{24}$ elements of order 5: 5-cycles $(1\underbrace{* * * *}_{2,3,4,5})$

$\binom{5}{4} \times 3! = 5 \times 6 = 30$

$\binom{5}{4}$: how many ways to choose i, j, k, l

$\underline{20}$ elements of order 6: $(i\ j\ k)(l\ m)$

$120 = |S_5|$

$(1\,2\,3)(4\,5) \in S_5$ has order 6

$(1\ 2\ 3\ 4)(5\ 6) \in S_6$ has order 4

If $\alpha \in S_n$ is written as a product of disjoint cycles, then its order is the least common multiple of the lengths of its cycles.

$(1\,2\,3)(4\,5\,6\,7\,8)$ has order 15

$(1\,2\,3)(4\,5\,6\,7\,8\,9)$ 6