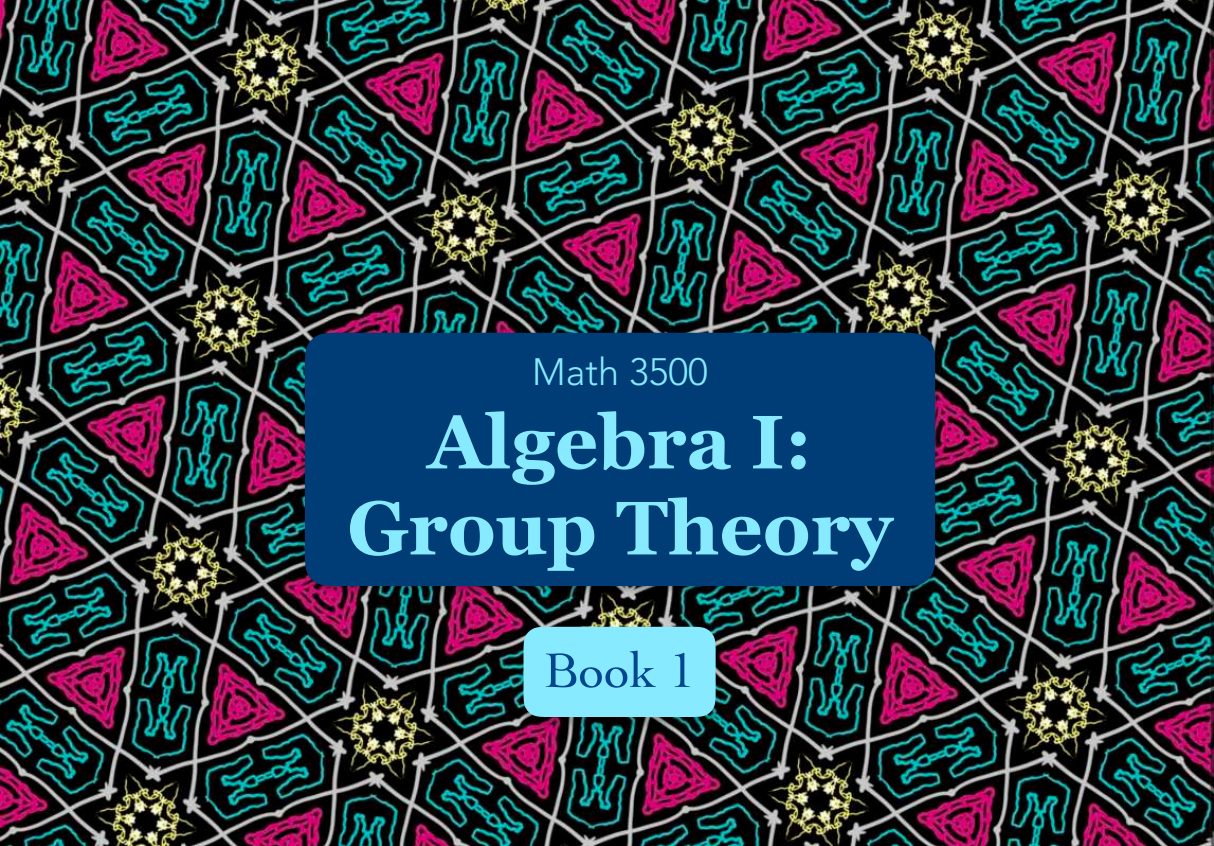

Math 3500
Algebra I:
Group Theory
Book 1

Symmetry group of a square  :

$G = \{I, R, R^2, R^3, H, V, D, D'\}$

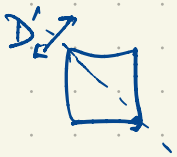
R = counter-clockwise rotation about center by 90°

R^2 = 180° rotation about center

R^3 = 270° counterclockwise rotation = 90° clockwise rotation

$R^4 = I$

D = reflection 

D' = 

H = 

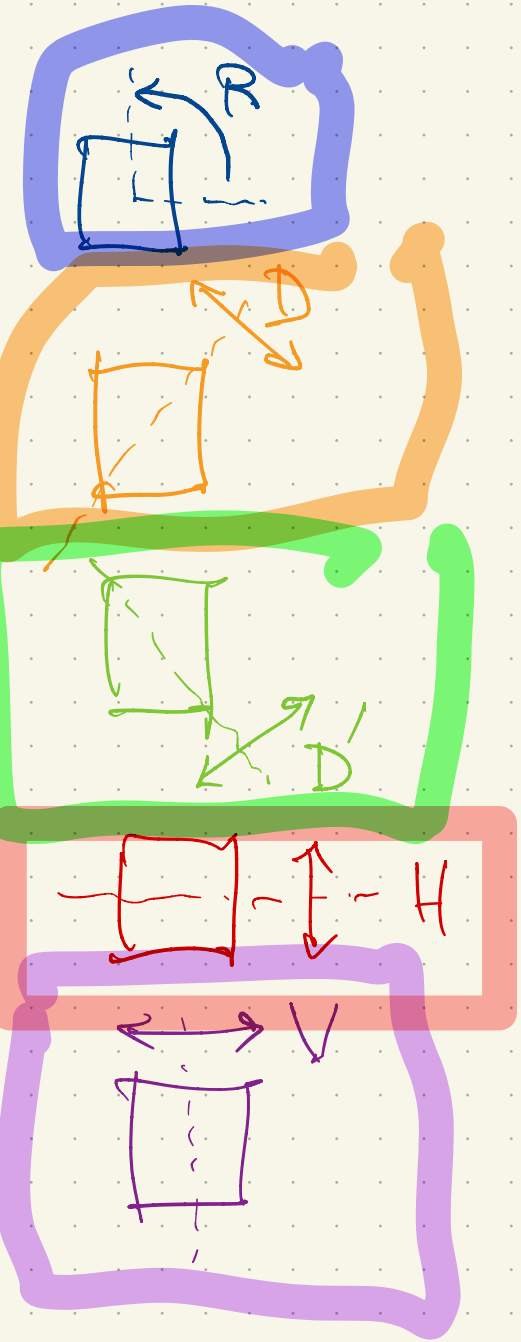
V = 

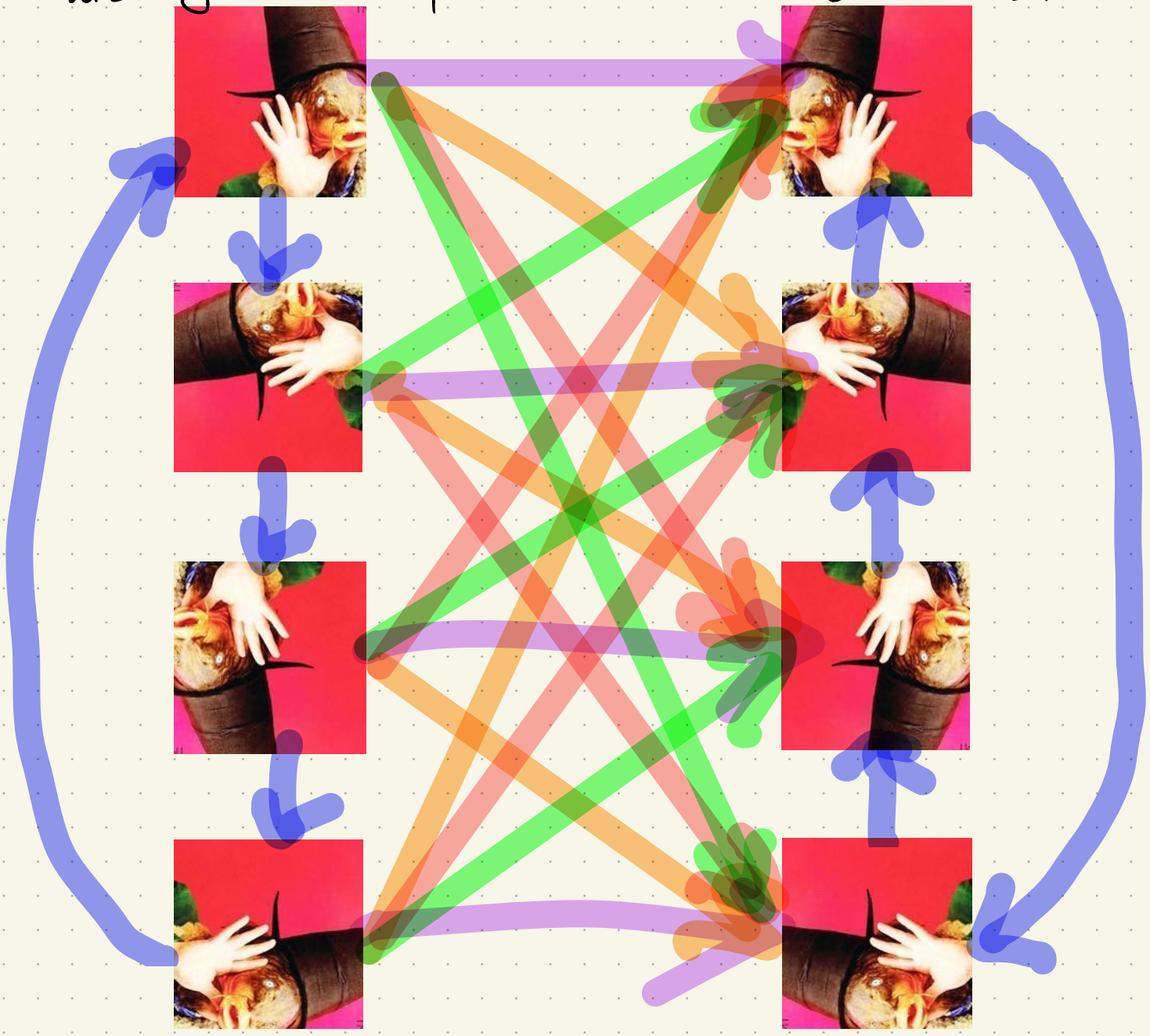
Symmetry group of square $G = \{I, R, R^2, R^3, D, D', H, V\}$

Group elements are transformations/functions/maps/mappings/arrows (not the images/squares on which the group elements act).

Virtual symmetries reverse orientation; (eg. reflections)

direct symmetries preserve orientation. (eg. rotations)



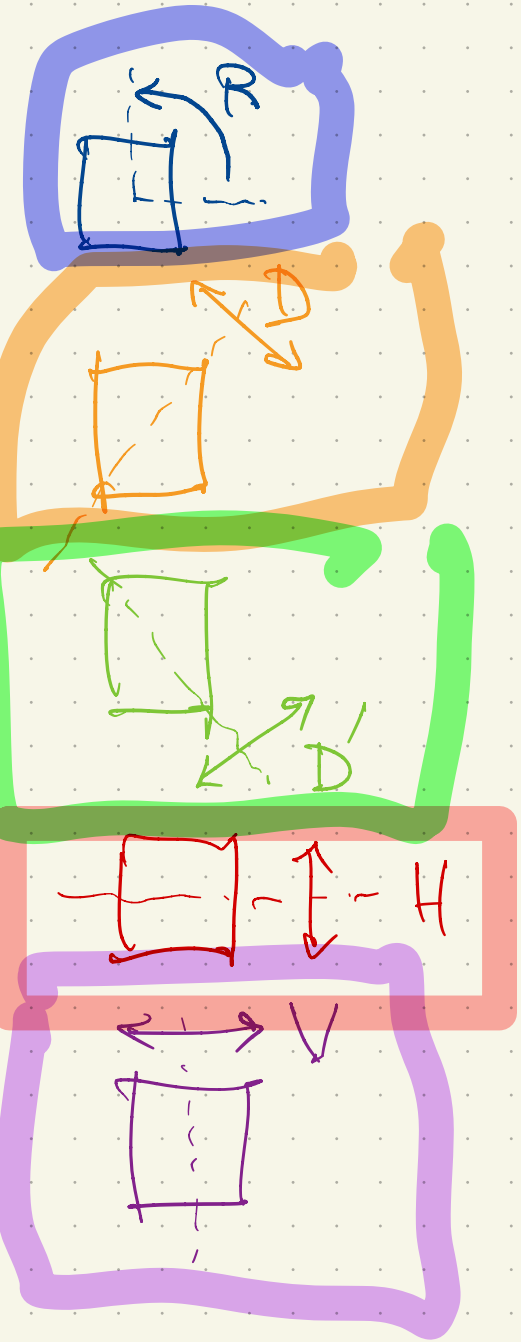


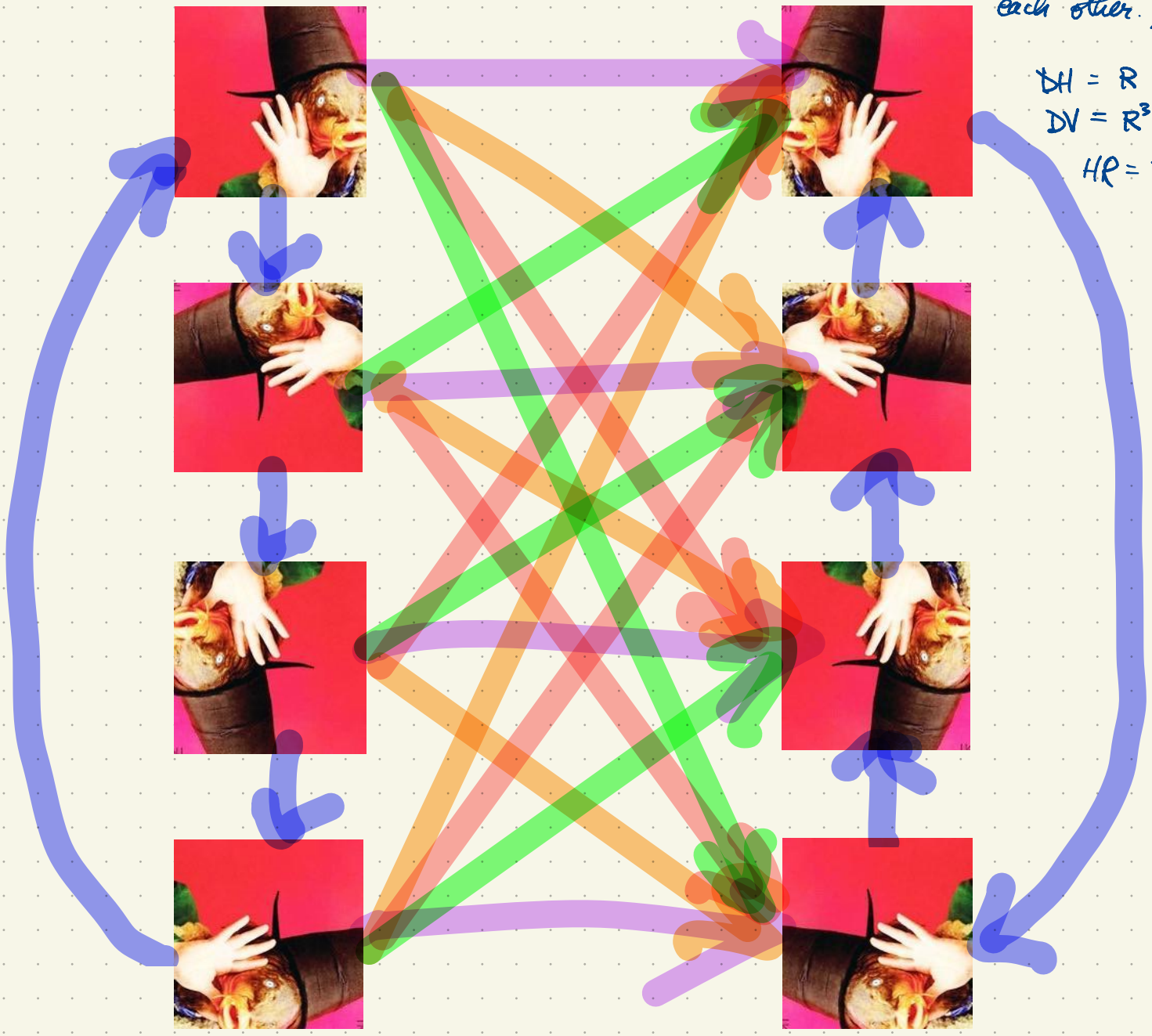
Symmetry group of square $G = \{I, R, R^2, R^3, D, D', H, V\}$

Composition (right-to-left): $RD = V$, $DR = H$, $HV = R^2$, $VH = R^2$

Note: H and V commute (ie. $HV = VH$) but R and D do not commute ($RD \neq DR$).

We say that G is nonabelian because its elements do not all commute with each other. (A group is abelian iff all its elements commute with each other.)





$DH = R$

$DV = R^3$

$HR = D'$

The multiplication table of G:

$G = \{I, R, R^2, R^3, D, D', H, V\}$ is the dihedral group of order 8.

The order of a group G is $|G|$ = number of elements in G.

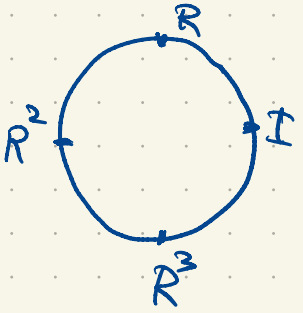
G has five elements of order 2: D, D', H, V, R^2;
two elements of order 4: R, R^3;
one element of order 1: I.

	I	R	R^2	R^3	D	D'	H	V
I	I	R	R^2	R^3	D	D'	H	V
R	R	R^2	R^3	I	V	H	D	D'
R^2	R^2	R^3	I	R	D'	D	V	H
R^3	R^3	I	R	R^2	H	V	D'	D
D	D	H	D'	V	I	R^2	R	R^3
D'	D'	V	D	H	R^2	I	R^3	R
H	H	D'	V	D	R^3	R	I	R^2
V	V	D	H	D'	R	R^3	R^2	I

$DR^2 = DR \cdot R = HR = D'$

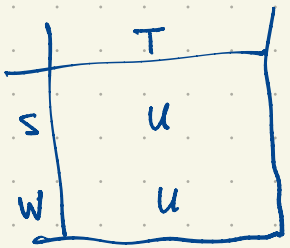
$D'R^2 = D'R \cdot R = VR = D$

$\langle R \rangle = \{I, R, R^2, R^3\}$



The (i,j) entry (i.e. row i, column j) indicates the i^{th} element "times" the j^{th} element.

In the multiplication table, each group element appears exactly once in each row and column.



$\Rightarrow \quad ST = U = WT \Rightarrow \quad ST \cdot T^{-1} = WT \cdot T^{-1} \Rightarrow \quad S = W$

Associativity holds!

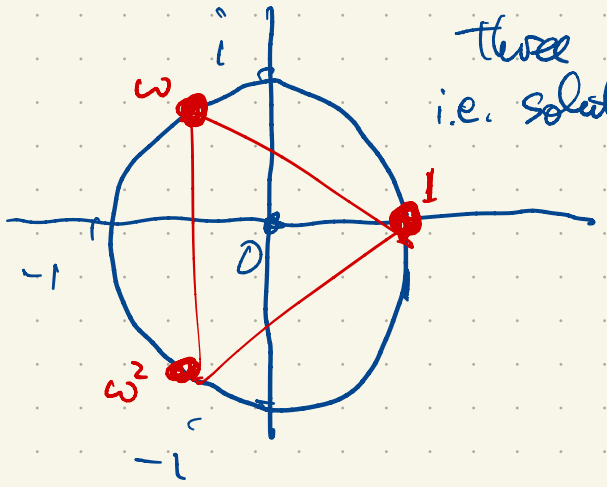
$f \circ (g \circ h) = (f \circ g) \circ h$

$f(g(h(x)))$

Eg. $\{1, \omega, \omega^2\}$, $\omega = \frac{-1+i\sqrt{3}}{2} = e^{i2\pi/3}$

↑ omega ≠ w

x	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω



three cube roots of unity in $\mathbb{C}$:
i.e. solutions of $x^3 = 1$, $x \in \mathbb{C}$.

$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$

$\mathbb{Q}$ = {rational numbers} $= \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, \ldots\}$

$\{1, \omega, \omega^2\}$ is a group of order 3 having two elements of order 3: ω, ω^2 and one element of order 1: 1.

Any group of order 3 is cyclic: it must have the form $\{1, g, g^2\}$, $g^3 = 1$.

	1	g	h
1	1	g	h
g	g	h	1
h	h	1	g

A cyclic group is a group generated by one element i.e.

$G = \{\ldots g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots\}$

$= \{g^k : k \in \mathbb{Z}\}$

i.e. consists of all powers of $g \in G$.

$\langle g \rangle$ = group generated by g

$= \{g^k : k \in \mathbb{Z}\}$

$g^k g^l = g^{k+l}$ for all $k, l \in \mathbb{Z}$

$g^0 = 1$

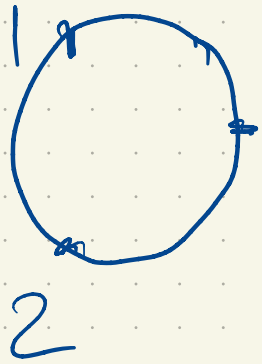
If $G = \{1, g, h\}$ is a group then $g^2 = h$ so $G = \{1, g, g^2\}$.

(the cyclic group of order 3)

	1	g	g^2
1	1	g	g^2
g	g	g^2	1
g^2	g^2	1	g

Note: The order of any group element $g \in G$ is $|\langle g \rangle| = |g|$

Eg. $\mathbb{Z}/3\mathbb{Z} = \{\text{integers mod } 3\} = \{0, 1, 2\}$ $2+2=1$ in this group with identity element 0.



+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$-1 = 2$
$-2 = 1$
$-0 = 0$

$1-2 = 2$

×	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

These two groups of order 3 are essentially the same as seen by their Cayley table (addition table and multiplication table respectively). More precisely, the groups $(\mathbb{Z}/3\mathbb{Z}, +)$ is isomorphic to $(\langle\omega\rangle, \times)$ i.e. $(\mathbb{Z}/3\mathbb{Z}, +) \cong (\langle\omega\rangle, *)$.

this means there is a bijection $\phi: \mathbb{Z}/3\mathbb{Z} \to \langle\omega\rangle$ satisfying $\phi(i+j) = \phi(i)\phi(j)$ for all $i, j \in \mathbb{Z}/3\mathbb{Z}$.

addition in $\mathbb{Z}/3\mathbb{Z}$

multiplication in $\mathbb{C}$ or in $\langle\omega\rangle$.

+	j
i	i+j

$\longrightarrow$

×	$\phi(j)$
$\phi(i)$	$\phi(i)\phi(j)$

Some infinite groups:

$(\mathbb{R}, +)$ is a group.

Identity element 0: $0 + a = a$

Inverses: inverse of $a \in \mathbb{R}$ is $-a$ since $a + (-a) = 0$

Associativity: $(a+b)+c = a+(b+c)$

(By the way, in this group, the identity 0 has order 1; every nonidentity element has infinite order. This group is abelian since $a+b = b+a$ for all $a, b \in \mathbb{R}$.)

$(\mathbb{R}, \times)$ is not a group. Here the identity element is $1 \in \mathbb{R}$ since $1a = a$.

In general $a \in \mathbb{R}$ does not have an inverse.

If $a \neq 0$ then $a^{-1} = \frac{1}{a}$ is the inverse of a since $\frac{1}{a} \cdot a = 1 =$ identity.

But $0 \in \mathbb{R}$ has no multiplicative inverse: $\square \cdot 0 = 1$ has no solution in $\mathbb{R}$.

The associative law holds for multiplication in $\mathbb{R}$: $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$

Fixing the previous example: one idea

$\mathbb{R}^\times = \{a \in \mathbb{R} : a \neq 0\} = \{$all nonzero real numbers$\} = \underbrace{(-\infty, 0)}_{\text{negative reals}} \cup \underbrace{(0, \infty)}_{\text{positive reals}}$

Multiplication in $\mathbb{R}$ is often written using '$\times$' or '$\cdot$' or (2) e.g.

$2 \times 3 = 6$

$2 \cdot 3 = 6$

$ab = a \times b = a \cdot b$

(juxtaposition; times symbol; dot)

"times" symbol.

$(\mathbb{R}^\times, \times)$ is a multiplicative group

identity: 1

inverses: $a^{-1} = \frac{1}{a}$ is the inverse of $a \in \mathbb{R}^\times$

associativity: $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}^\times$.

(By the way, this group is abelian and infinite.)

1 has order 1 (as in any group).

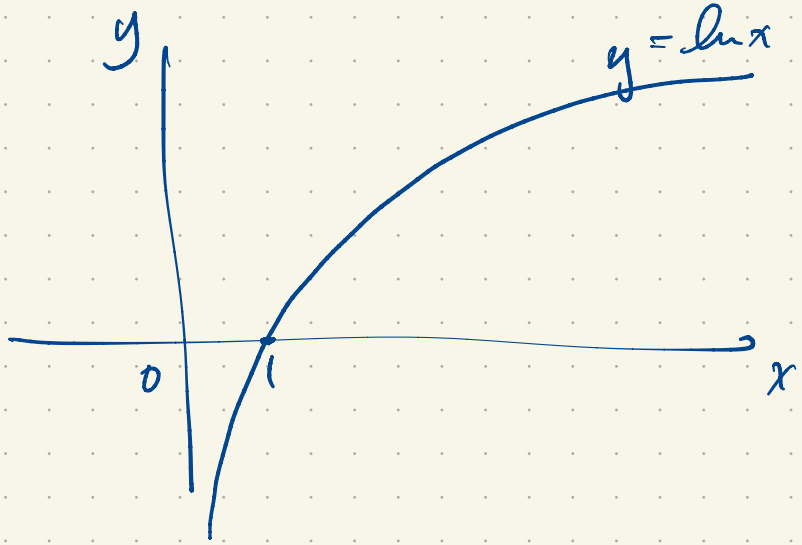
-1 has order 2

Any $a \in \mathbb{R}^\times$ other than ± 1 has infinite order.

$((0,\infty), \times)$ is a group

1 is the identity (order 1). All other elements have infinite order.

$(\mathbb{R}, +)$ and $((0,\infty), \times)$ are isomorphic groups.



$ab = ?$

$\ln a \longleftarrow a$

$\ln b \longleftarrow b$

$\ln a + \ln b \longrightarrow e^{\ln a + \ln b} = e^{\ln a} \cdot e^{\ln b} = ab$

ln is a group isomorphism from $((0,\infty), \times)$ to $(\mathbb{R}, +)$.
The inverse function, $\exp(a) = e^a$, is a group isomorphism from $(\mathbb{R}, +)$ to $((0,\infty), \times)$.

A group is a set G together with a binary operation $*$ on G such that
(i) there is an element $e \in G$ such that $e * x = x * e = x$ for every $x \in G$;
(i.e. e is an identity element for '$*$');
(ii) for every $x \in G$ there is an element $y \in G$ such that $x * y = e = y * x$
(i.e. y is an inverse element for x);
(iii) for all $x, y, z \in G$, $(x * y) * z = x * (y * z)$ (i.e. '$*$' is associative).

Note: a binary operation '$*$' on G is a function $G \times G \longrightarrow G$, $(x, y) \longmapsto \underbrace{*(x,y)}_{\text{written as } x*y}$

Eg. exponentiation is a binary operation on $(0,\infty) = \{\text{positive real numbers}\}$.

$(a,b) \longmapsto a^b$.

Is this a group operation?

If $e \in (0,\infty)$ is an identity element for this operation then $e^a = a$ and $a^e = a$ for all $a \in (0,\infty)$. Then $2^e = 2$ so $e = 1$. But then $1^a = a$, i.e. $1^2 = 2$. This does not hold. So exponentiation is a binary operation on $(0,\infty)$ which has no identity element.

So $((0,\infty), \text{exponentiation})$ is not a group.

"$\wedge$" $\quad a \wedge b = a^b$

Moreover, exponentiation is not associative. Check: does

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$? No!

eg. $(3\wedge 3)\wedge 2 \overset{?}{=} 3 \wedge (3 \wedge 2)$

$27^2 = 729 \qquad 3^9 = 19683$

$(2\wedge 2)\wedge 2 \overset{?}{=} 2 \wedge (2\wedge 2)$

$4^2 = 16 \qquad 2^4 = 16$

Exponentiation is not a binary operation on $\mathbb{R}$.

$(-1)^{1/2} = ?$ $\quad i$? $\quad -i$?

$0^{-1} = ?$ undefined!

Subtraction is a binary operation on $\mathbb{R}$: $(a,b) \longmapsto a-b$.

If $e \in \mathbb{R}$ is a binary operation for subtraction then $a - e = a = e - a$ for all $a \in \mathbb{R}$.

$2 - e = 2$, $\quad e - 2 = 2$ No value $e \in \mathbb{R}$ satisfies both of these conditions.

Is subtraction associative? Does $(x-y)-z = x-(y-z)$ for all $x,y,z \in \mathbb{R}$?

Try $(3-4)-7 = -1-7 = -8$

$3-(4-7) = 3-(-3) = 6$ } Not equal. Subtraction is nonassociative.

Eg. let $n \geq 1$ (a positive integer) and let $G = \{n \times n$ invertible $n \times n$ matrices with real entries$\} = GL_n(\mathbb{R})$ (general linear group of $n \times n$ real matrices).

Note: an $n \times n$ matrix A is in G iff $\det A \neq 0$. $\quad GL_1(\mathbb{R}) = \mathbb{R}^\times$

$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is the identity element.

eg. $n=2$: the group $G = GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$. Identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

eg. $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \in G$

$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$A^3 = AA^2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

$A^4 = A^3A = -A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$

$A^5 = A^3A^2 = -A^2 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$A^6 = A^3A^3 = (-I)(-I) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so A has order 6.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ shear

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$

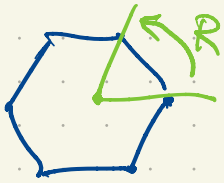
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has infinite order: there is no positive integer n for which $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = I$.

Most elements in G have infinite order.

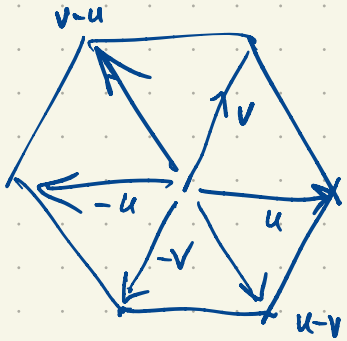
Most elements in G do not commute

For every positive integer n, there exist element(s) in G of order n. How do we construct them?

A regular hexagon has twelve symmetries forming a dihedral group of order 12.
Six of these are rotational symmetries and six reflective symmetries



R (a 60° rotation about the centre with an (x,y) coordinate system also centered at this point) gives an element of $G = GL_2(\mathbb{R})$ of order 6.
I'll choose a basis $\{u, v\}$ for $\mathbb{R}^2$ as shown:
(this is not the standard basis)



$Ru = v$

$Rv = -u+v = v-u$

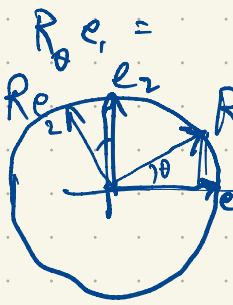
$R(au+bv) = aRu + bRv = av + b(v-u)$
$= -bu + (a+b)v$

Writing vectors in $\mathbb{R}^2$ as column vectors of coefficients with respect to our basis.

$R: \begin{bmatrix} a \\ b \end{bmatrix} \longmapsto \begin{bmatrix} -b \\ a+b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

So this matrix represents R and it has order 6.

What if we choose the standard basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?
Now let R_θ be the counter-clockwise rotation about the origin by angle θ.



$R_\theta e_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $R_\theta e_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

R_θ is represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

For $\theta = 60° = \frac{\pi}{3}$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$R_{\frac{\pi}{3}} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$
is an element of $GL_2(\mathbb{R})$ of order 6.

In fact $R_\theta \in SL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc = 1 \right\}$. This is a subgroup of $GL_2(\mathbb{R})$:
Just like $(0,\infty) \leq \mathbb{R}^\times = (-\infty,0) \cup (0,\infty)$.
$SL_2(\mathbb{R}) \leq GL_2(\mathbb{R})$.

	1	a	b	c
1	1	a	b	c
a	a	c	1	b
b	b	1	c	a
c	c	b	a	1

or

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

or

	1	a	b	c
1	1	a	b	c
a	a	b	c	1
b	b	c	1	a
c	c	1	a	b

cyclic group of order 4

$G = \langle a \rangle = \{1, a, a^2, a^3\}$ ($a^2 = b$, $a^3 = c$)

or

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	a	1
c	c	b	1	a

cyclic group of order 4

$\langle b \rangle = \{1, b, b^2, b^3\}$ ($b^2 = a$, $b^3 = c$)

Groups of order 4:

Let G be a multiplicative group of order 4 with elements $1, a, b, c$ where 1 is the identity.

$(ab)c = cc = 1$

$a(bc) = aa = 1$

$(ab)b = cb = a$

$a(bb) = a1 = a$

This group has 1 element of order 1;
3 elements ... 2.
(the Klein four-group)

Rewrite $1 \to 1$, $a \to b$, $b \to a$, $c \to c$

	1	a	b	c
1	1	a	b	c
a	a	b	c	1
b	b	c	1	a
c	c	1	a	b

$\cong$

	1	b	a	c
1	1	b	a	c
b	b	a	c	1
a	a	c	1	b
c	c	1	b	a

$=$

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	a	1
c	c	b	1	a

The cyclic group of order 4 has 1 element of order 1;
1 " " " 2;
2 elements " " 4.

There are essentially two groups of order 4:
a cyclic group of order 4
and the Klein four-group.

Suppose G is a multiplicative group.

Every element $g \in G$ generates a cyclic subgroup $\langle g \rangle = \{\text{all powers of } g\}$

$= \{g^k : k \in \mathbb{Z}\} = \{\dots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \dots\}$.

If $\langle g \rangle = G$ then G is cyclic.

If g has finite order n then $|g| = |\langle g \rangle| = n$. This requires some thought to fully understand. If $|g| = n$ then $g, g^2, \dots, g^{n-1}$ are non-identity elements in G whereas $g^n = 1$.

$g^k g^l = g^{k+l}$, $(g^k)^l = g^{kl}$ for all $k, l \in \mathbb{Z}$.

$g^2 g^3 = (gg)(ggg) = g^5$

$g^{-2} g^5 =$

Why don't we write $\frac{1}{A}$ for $A \in GL_n(\mathbb{R})$? We do write $\frac{1}{5}$ for $5 \in \mathbb{R}^\times$.

In $\mathbb{R}^\times$, $\frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{b}$

Does $(AB)^{-1} = A^{-1}B^{-1}$? No; $(AB)^{-1} = B^{-1}A^{-1}$

In any multiplicative group, $(gh)^{-1} = h^{-1}g^{-1}$. Why? $(gh)^{-1}(gh) = 1$

$(h^{-1}g^{-1})(gh) = h^{-1}(g^{-1}g)h = h^{-1}1h = h^{-1}h = 1$.

$(gh)(h^{-1}g^{-1}) = 1$.

If G is any group with operation $*$ and identity e, then G has a unique identity e. Why? If $e' \in G$ is also an identity then

$$e * x = x * e = x$$
$$e' * x = x * e' = x \quad \text{for all } x \in G.$$

Then $e = e * e' = e'$.