Algebra I

Group Theory

Book 3

A matrix in $GL_2(\mathbb{R})$ is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ iff it has trace 0 and determinant -1.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{R})$ then A has characteristic polynomial $f(x) = \det(xI - A) = \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$$= \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} = (x-a)(x-d) - bc = x^2 - \underbrace{(a+d)}_{\operatorname{tr} A} x + \underbrace{(ad-bc)}_{\det A}.$$

Some books define the characteristic polynomial of A as $\det(A - xI) = (-1)^n \underbrace{\det(xI - A)}_{\substack{\text{monic: its leading} \\ \text{term is } x^n.}}$

Cayley-Hamilton Theorem (look it up in any linear algebra book)
If $f(x)$ is the characteristic polynomial of an $n \times n$ matrix A, then $f(A) = 0$.

In the 2×2 case, $A^2 - (\operatorname{tr} A)A + (\det A)I = 0$ holds as we compute here:

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

$$A^2 - (\operatorname{tr} A)A + (\det A)I = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a^2+bc-(a+d)a+(ad-bc) & ab+bd-(a+d)b \\ ac+cd-(a+d)c & bc+d^2-(a+d)d+(ad-bc) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If $A \in GL_2(\mathbb{R})$ has trace 0 and determinant -1 then it satisfies $A^2 - \underline{0}A - 1I = 0$ so $A^2 = I$
so in the group $GL_2(\mathbb{R})$, A has order ~~1 or~~ 2. ($\operatorname{tr} I = 2$, not 0)

$f(x) = \det(xI - A)$ may or may not be the smallest degree polynomial that has A as a root.
The minimal polynomial of A, $m(x)$, is the monic polynomial of smallest degree satisfying $m(A) = 0$.

Facts (see a linear algebra book):

- Roots of $f(x)$ are eigenvalues of A.
- $m(x)$ divides $f(x)$ i.e. $f(x) = h(x)m(x)$ for some monic polynomial $h(x)$ (often $h(x) = 1$, $m(x) = f(x)$).
- Every eigenvalue of A is a root of $m(x)$.

Theorem Let $A \in GL_2(\mathbb{R})$. Then the following are equivalent:

(i) $\operatorname{tr} A = 0$, $\det A = -1$

(ii) A has order 2 but $A \neq -I$.

(iii) A is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We have proved (i) $\Rightarrow$ (ii). And (iii) $\Rightarrow$ (i) is easy. Assume $A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ for some $M \in GL_2(\mathbb{R})$.

Then $\operatorname{tr} A = \operatorname{tr}\left(M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}\right) = \operatorname{tr}\left(M^{-1}M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \operatorname{tr}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0$.

$\operatorname{tr} AB = \operatorname{tr} BA$ if A is $m \times n$, B is $n \times m$ (short proof: see linear algebra. Both equal to $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ji}$)

$\det A = \det M \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\det M^{-1}}_{(\det M)^{-1}} = -1.$

$MM^{-1} = I$

$\det(M)\underbrace{\det(M^{-1})}_{1/\det M} = \det I = 1$

We must prove (ii) $\Rightarrow$ (iii). If A has order 2 then $A^2 = I$, $A \neq I$. A is a root of $x^2 - 1 = (x+1)(x-1)$

so the minimal poly. of A divides $x^2 - 1$: $m(x) = x^2 - 1$ or $x+1$ or $x-1$ or 1.

If $m(x) = 1$ then $m(A) = I = 0$. No!.

If $m(x) = x-1$ then $m(A) = A - I = 0$ then $A = I$ (No! I has order 1, not order 2)

If $m(x) = x+1$ then $m(A) = A + I = 0$ so $A = -I$ (No! by assumption).

So $m(x) = x^2 - 1$ divides $f(x)$, so $f(x) = x^2 - 1$. $\Rightarrow \operatorname{tr} A = 0$, $\det A = -1$. $\Rightarrow$ (i) holds

So ± 1 are eigenvalues of A. Let u, v be eigenvectors corresponding to $1, -1$ i.e. $Au = u$, $Av = -v$.

Let $M = [u \mid v]$ (2×2 matrix having u, v as columns)

$AM = [Au \mid Av] = [u \mid -v] = [u \mid v]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ i.e. (iii) holds. $\square$

There are two conjugacy classes of elements of order 2 in $G = GL_2(\mathbb{R})$:

- $\{-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\}$ is in a class by itself since $-I \in Z(G)$
- All matrices conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e. all matrices with trace 0 and determinant -1.
  This includes $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$, $a \in \mathbb{R}$

Consider the dihedral group G of order 8 (the symmetry group of a square), so $|G| = 8$.
Let's pick generators x, y for G where x is an element of order 4 and y is a reflection (order 2).

$G = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$, $yx = x^3y$ i.e. $yxy^{-1} = yxy = x^{-1} = x^3$,

$$\left.\begin{aligned} x^i \cdot x^j &= x^{i+j} \\ x^i \cdot x^j y &= x^{i+j} y \\ x^i y \cdot x^j &= x^{i-j} y \\ x^i y \cdot x^j y &= x^{i-j} \end{aligned}\right\}$$

"If you move y past x^i, it inverts $x^i \mapsto x^{-i}$"

$$x^i y x^j y = x^i \underbrace{(yxy)(yxy)\cdots(yxy)}_{(yxy)^j} = x^i (x^j)^{-1} = x^i x^{-j} = x^{i-j}$$

Presentation for G : $G = \langle \underbrace{x, y}_{\text{generators}} : \underbrace{x^4 = y^2 = 1, \; yx = x^3y}_{\text{relations}} \rangle$

$x^2 \cdot y = x^2 y$

$yx^2 = x^{-2} y = x^2 y$ — $i=0$, $j=2$ in the rule $x^i y \cdot x^j = x^{i-j} y$

g	$\lvert g\rvert$	$C_G(g)$	
1	1	G,	$\lvert G\rvert = 8$
x	4	$\langle x\rangle$	$\lvert\langle x\rangle\rvert = 4$
x^3	4	$\langle x\rangle$	$\lvert\langle x\rangle\rvert = 4$
x^2	2	G,	$\lvert G\rvert = 8$
y	2	$\langle x^2, y\rangle$	$\lvert\langle x^2, y\rangle\rvert = 4$
x^2y	2	$\langle x^2, y\rangle$	$\lvert\langle x^2, y\rangle\rvert = 4$
xy	2	$\langle x^2, xy\rangle$	$\lvert\langle x^2, xy\rangle\rvert = 4$
x^3y	2	$\langle x^2, xy\rangle$	$\lvert\langle x^2, xy\rangle\rvert = 4$

Centralizer of $g \in G$:

$$C_G(g) = \{x \in G : xg = gx\}$$

$Z(G) = \langle x^2 \rangle = \{1, x^2\}$.

$C_G(y) = \{1, x^2, y, x^2y\}$ is a Klein four-group

$\mathcal{O}(x) = \{x, x^3\}$

$\mathcal{O}(1) = \{1\}$

$\mathcal{O}(x^2) = \{x^2\}$

$C_G(xy) = \{1, x^2, xy, x^3y\}$ is a Klein four-group

If $\mathcal{O}(g)$ is the conjugacy class of $g \in G$ then $|\mathcal{O}(g)||C_G(g)| = |G|$. eg. $1 \times 8 = 8$, $2 \times 4 = 4$.