Algebra I

Group Theory

Book 1

A *group* is a set G with a binary operation $*$ which has an identity element; the operation is associative; and every element has an inverse.

Eg. $\mathbb{R}$ = set of real numbers under addition '+'. Its identity element is 0.

$$\left.\begin{aligned} 0+x&=x\\ (x+y)+z&=x+(y+z)\\ x+(-x)=0&=(-x)+x \end{aligned}\right\} \quad \text{for all } x,y,z\in\mathbb{R}$$

$(\mathbb{R},+)$ is a group.

$(\mathbb{R},\times)$ (real numbers under multiplication) is almost but not quite a group. (0 does not have an inverse). 1 is the identity.

$\mathbb{R}^\times$ = {all nonzero real numbers} $=\{a\in\mathbb{R} : a\neq 0\}$ is a group under multiplication.

$$\left.\begin{aligned} 1a&=a\\ (ab)c&=a(bc)\\ a\cdot a^{-1}&=a^{-1}\cdot a=1 \qquad a^{-1}=\tfrac{1}{a} \end{aligned}\right. \quad \text{for all } a,b,c\in\mathbb{R}^\times.$$

$(\mathbb{R}^\times,\times)$ is a group.

$\mathbb{R}$ with the operation $x*y=x+y+7$. This is a group $(\mathbb{R},*)$. For all $x,y,z\in\mathbb{R}$,

$$(x*y)*z=(x+y+7)+z+7=x+y+z+14=x+(y+z+7)+7=x*(y*z)$$

so $(\mathbb{R},*)$ is associative. Note that $-7\in\mathbb{R}$ is an identity element since

$$\left.\begin{aligned} -7*x&=(-7)+x+7=x\\ \text{and}\quad x*(-7)&=x+(-7)+7=x \end{aligned}\right\} \quad \text{for all } x\in\mathbb{R}. \qquad \text{So } -7\in\mathbb{R} \text{ is an identity element for '}*\text{'}.$$

$$\left.\begin{aligned} (-x-14)*x&=(-x-14)+x+7=-7\\ x*(-x-14)&=x+(-x-14)+7=-7 \end{aligned}\right\} \quad \text{for all } x\in\mathbb{R}. \qquad \text{So } -x-14 \text{ is an inverse element for } x.$$

$(x*y)*z = x*(y*z)$
$\Leftrightarrow (x+y+7)+z+7 = x+(y+z+7)+7$
$\Leftrightarrow x+y+z+14 = x+y+z+14$

so $(\mathbb{R}, *)$ is associative.

$7=3$
$\Rightarrow 7-5 = 3-5$
$\Rightarrow 2 = -2$
$\Rightarrow (2)^2 = (-2)^2$
$\Rightarrow 4 = 4$

$(x*y)*z = (x+y+7)+z+7$
$= x+y+z+14$
$= x+(y+z+7)+7$
$= x*(y*z)$

$(\mathbb{Q}, +)$ is a group. $\qquad \mathbb{Q} = \{\text{rational numbers}\}$ $\qquad -\frac{5}{3} \in \mathbb{Q}$

$\frac{172}{100} = 1.72 \in \mathbb{Q}$

$\pi \notin \mathbb{Q}$

$\sqrt{2} \notin \mathbb{Q}$

$(\mathbb{Q}^\times, \times)$ is a group.

$\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\} = \{\text{all nonzero rational numbers}\}$

$(\mathbb{N}, +)$ is not a group

$\mathbb{N} = \{1,2,3,4,\cdots\} = \mathbb{Z}^{>0}$

$\mathbb{N}_0 = \{0,1,2,3,4,\cdots\} = \mathbb{Z}^{\geq 0}$

$\mathbb{Z} = \{\text{integers}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

$(\mathbb{Z}, +)$ is a group.

$\underset{\text{subgroup}}{(\mathbb{Z}, +) \leq} (\mathbb{Q}, +) \underset{\text{subgroup}}{\leq} (\mathbb{R}, +) \leq (\mathbb{C}, +)$; but $(\mathbb{R}^\times, \times)$ is not a subgroup $(\mathbb{R}, +)$ $\not\leq$ (although $\mathbb{R}^\times \underset{\text{subset}}{\subseteq} \mathbb{R}$)

In $\mathbb{R}^\times$, $2 \times 3 = 6$ but in $(\mathbb{R}, +)$, $2+3=5$

$GL_n(\mathbb{R}) = \{$ invertible $n\times n$ matrices with real entries $\}$ is the <u>general</u> <u>linear</u> <u>group</u>

$GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$GL_n(\mathbb{R})$ is a multiplicative group with identity $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$GL_n(\mathbb{R})$ is <u>not</u> commutative for $n \geq 2$.

$GL_1(\mathbb{R})$ is commutative.

$(G, *)$ is <u>Abelian</u> (abelian) if $x*y = y*x$ for all $x,y \in G$.

$GL_n(\mathbb{R})$ is abelian for $n=1$; nonabelian for $n \geq 2$. $\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 5 & 35 \end{bmatrix}$ whereas $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -4 & 38 \end{bmatrix}$.

$GL_1(\mathbb{R}) \cong \mathbb{R}^\times$ (these are isomorphic groups i.e. essentially the same group. Since $\mathbb{R}^\times$ is abelian, so is $GL_1(\mathbb{R})$.)

Function composition is associative: $(f\circ g)\circ h = f\circ(g\circ h)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"'] & Y \arrow[r, "g"] \arrow[rr, bend right, "f\circ g"'] & Z \arrow[r, "f"] & W \end{tikzcd}$$

If $x \in X$ then $h(x) \in Y$, $g(h(x)) \in Z$, $\underbrace{f(g(h(x)))}_{(f\circ g\circ h)(x)} \in W$.

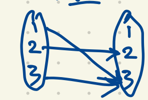
$f\circ g \neq g\circ f$

Because matrix multiplication is expressing the composition of linear transformations, it is associative but not necessarily commutative.

If X is any set, the bijections $X \xrightarrow{f} X$ (i.e. f one-to-one and onto) form a group under composition. This is the <u>symmetric group</u>

$$G = \mathrm{Sym}\,X = \{\text{bijections } X \to X\} = \{\text{permutations of } X\}.$$

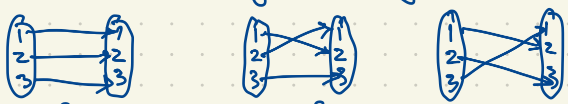
Not a bijection (neither one-to-one nor onto)



eg. $X = [3] = \{1,2,3\}$. (Notation: $[n] = \{1,2,3,\dots,n\}$.)

There are exactly $3! = 6$ bijections $[3] \to [3]$.

$n! = 1 \times 2 \times 3 \times \cdots \times n$ (n factorial) is the number of permutations of $[n]$.



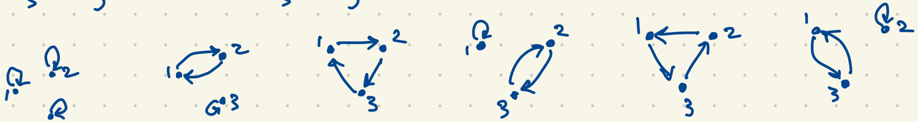
x	$f(x)$
1	1
2	2
3	3

x	$f(x)$
1	2
2	1
3	3

$|S_3| = 6$. S_3 is a nonabelian group of order 6.

S_3 is the smallest nonabelian group.

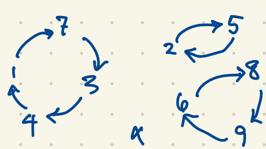
In S_3, $(12)(13) = (132)$
$(13)(12) = (123)$



$(\ )$ (12) (123) (23) (132) (13)

cycle notation for $\mathrm{Sym}[3] = S_3$
$= \{(\ ), (12), (13), (23), (123), (132)\}$ — $(12)(3)$

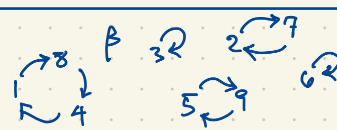
Eg. $n = 9$



α

n	$\alpha(n)$	$\beta(n)$	$\alpha\beta(n)$
1	7	8	9
2	5	7	3
3	4	3	4
4	1	1	7
5	2	9	6
6	8	6	8
7	3	2	5
8	9	4	1
9	6	5	2

β



$\beta = \underbrace{(7,2)}\underbrace{(4,1,8)}(3)(6)(5,9) = (1\,8\,4)(2\,7)(5\,9)$

$(7,2) = (2,7)$

$(4,1,8) = (1,8,4) = (8,4,1)$

$(3) = (\ )$

$\alpha = (1,7,3,4)(2,5)(6,8,9)$

$\alpha\beta = \alpha \circ \beta = (1,9,2,3,4,7,5,6,8) = (1734)(25)(689)(184)(27)(59)$

$\beta\alpha = \beta \circ \alpha = (129648573) = (184)(27)(59)(1734)(25)(689)$

If α, β are permutations then $\alpha\beta \neq \beta\alpha$ in general but they have the same cycle structure.

The <u>order</u> of a group G is $|G|$, the number of elements in the group. (finite or infinite)

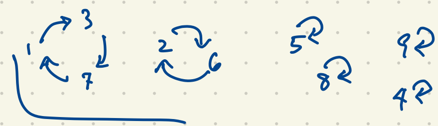
$|S_n| = n!$

$|GL_n(\mathbb{R})| = \infty$

S_n is nonabelian for $n \geq 3$.

$S_2 = \{(), (12)\}$ is abelian.

In S_n, disjoint cycles always commute, e.g. in S_9, $(137)(26) = (26)(137)$



If two permutations commute, must they have disjoint cycles?

$\alpha = (135)(246)$

$\beta = (12)(34)(56)$

Note: The two 3-cycles in α intersect with the three 2-cycles in β.

$\alpha\beta = (135)(246)(12)(34)(56) = (145236)$

$\beta\alpha = (12)(34)(56)(135)(246) = (145236)$