Algebra I

Group Theory

Book 2

Transpositions $(i\,j)$ are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd ; and vice versa.
A cycle of odd length is an even permutation;
" " " " even " " " odd " .

If α is a product of an even number of transpositions, then α is an even permutation.
" " " " " " " odd " " " " " " " " odd " .

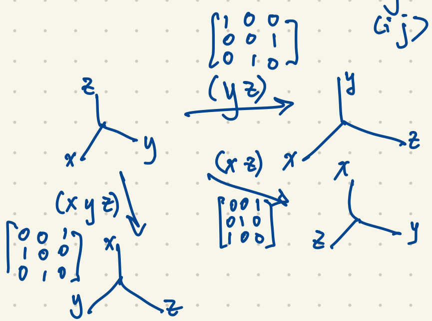
Permutations in S_5 :

Even		Odd	
$()$	1	$(i\,j)$	10
$(i\,j\,k)$	20	$(i\,j\,k\,l)$	30
$(i\,j\,k\,l\,m)$	24	$(i\,j\,k)(l\,m)$	20
$(i\,j)(k\,l)$	15		
	60		60

$|S_5| = 120$

$A_5 = \{$ even permutations in $S_5 \}$

$|A_5| = 60$



An even permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-preserving transformation.
An odd permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-reversing transformation.
If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then

$$\det T \begin{cases} = 0 & \text{if } T \text{ is not invertible} \\ > 0 & \text{" " preserves orientation} \\ < 0 & \text{" " " reverses " "} \end{cases}$$

A permutation $\alpha \in S_n$ can be expressed as a product of transpositions.
If α is a product of an even number of transpositions, then α is even.
" " " " " " odd " " " " " " " " " " " " " " odd.

In S_3:
$(13)(12)(13)(23)(23)(23)(12)(23) = (1\,2\,3)$ says (123) is an even permutation.

$S_3 \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\rangle \cong$ dihedral group of order 6 (symmetry group of an equilateral triangle)

n	no. of groups of order n up to isomorphism
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	5

Groups of order 2

$S_2 \cong \{0, 1\} \bmod 2$ under addition $\cong \langle -1 \rangle$ under multiplication

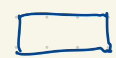
$\circ$	$()$	(12)
$()$	$()$	(12)
(12)	(12)	$()$

$+$	0	1
0	0	1
1	1	0

$\cdot$	1	-1
1	1	-1
-1	-1	1

Cayley tables of groups of order 2 all "look the same"

has a cyclic symmetry group of order 4



has an abelian symmetry group of order 4 which is not cyclic (the Klein four-group)

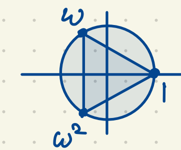
<u>Theorem</u> Any two groups of prime order are isomorphic; they are cyclic of order p.

Eg. $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ (under addition mod 3) is isomorphic to $A_3 = \langle (123) \rangle = \{(), (123), (132)\}$ and $\{1, \omega, \omega^2\}$ under multiplication, $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∘	()	(123)	(132)
()	()	(123)	(132)
(123)	(123)	(132)	()
(132)	(132)	()	(123)

·	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω



We say two groups G, H are isomorphic $(G \cong H)$ if there exists a bijection $\phi : G \to H$ such that $\phi(xy) = \phi(x)\phi(y)$

operation in G — operation in H

G

	y
x	xy

$\xrightarrow{\phi}$

H

	$\phi(y)$
$\phi(x)$	$\phi(xy)$

An isomorphism $\phi : \mathbb{Z}/3\mathbb{Z} \to A_3$ is a bijection satisfying $\phi(x+y) = \phi(x) \circ \phi(y)$

An isomorphism $\phi : \underbrace{\mathbb{R}}_{\text{under addition}} \longrightarrow \underbrace{(0,\infty)}_{\text{under multiplication}}$, $\phi(x+y) = \phi(x)\phi(y)$ is defined by $\phi(x) = e^x$

$e^{x+y} = e^x \cdot e^y$.

(subgroup of $\mathbb{R}^\times = (-\infty, 0) \cup (0, \infty)$)

$\ln = \phi^{-1} : (0,\infty) \to \mathbb{R}$

$\mathbb{R} \not\cong \mathbb{R}^\times$

since $\mathbb{R}$ (reals under addition) has only one element of finite order whereas $\mathbb{R}^\times$ has two elements of finite order: ± 1.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}/3\mathbb{Z}$

is isomorphic to

*	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\phi(0) = c$
$\phi(1) = a$
$\phi(2) = b$

*	c	a	b
c	c	a	b
a	a	b	c
b	b	c	a

or $\phi(0) = c$
$\phi(1) = b$
$\phi(2) = a$

*	c	b	a
c	c	b	a
b	b	a	c
a	a	c	b

+	0
0	0

Every group of order 1 is isomorphic to

·	1
1	1

(trivial group $\{1\}$)

" " " — — " " 2 " — — " " $\mathbb{Z}/2\mathbb{Z}$

+	0	1
0	0	1
1	1	0

	c
a	ac
b	bc

If $ac = bc$ then multiply both sides by c^{-1} on the right to get

$$(ac)c^{-1} = (bc)c^{-1}$$
$$a(cc^{-1}) = b(cc^{-1})$$
$$a1 = b1$$
$$a = b$$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Every group of order 3 is cyclic (isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under addition).

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Two cases: either all non-identity elements of G have order 2, or G has an element not of order 2.

<u>Theorem</u>: There are exactly two groups of order 4 up to isomorphism: the Klein four-group and the cyclic group of order 4.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

cyclic group of order 5

$$\langle a \rangle = \{e, a, \underset{b}{a^2}, \underset{c}{a^3}, \underset{d}{a^4}\}$$

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	a	e
c	c	d	e	b	a
d	d	b	a	e	c

is not a group!

It is a quasigroup, in fact since it has an identity e, it is a loop (its Cayley table is a Latin square: each row/column is a permutation of e, a, b, c, d).

This loop is not associative

e.g. $(ca)d = dd = c$

$c(ad) = cb = e$

c is a left inverse for b ($cb = e$) but not a right inverse for b ($bc = a$).

<u>Theorem</u> If every non identity element of a group G has order 2, then G is abelian.

<u>Proof</u> (Note: $x^2 = e$ = identity for every $x \in G$.)

Let $x, y \in G$. Then $(xy)^2 = xyxy = e$ so

$$yx = \underbrace{x(x}_{x^2=e}yx\underbrace{y)y}_{y^2=e} = xey = xy.$$

$\square$

In such groups, $x^{-1} = x$ for all $x \in G$.

Shoe-Sock Theorem

In every group G, with identity 1, for $x, y \in G$ we have $(xy)^{-1} = y^{-1}x^{-1}$.

Proof $(y^{-1}x^{-1})(xy) = y^{-1} 1 y = 1$ and $(xy)(y^{-1}x^{-1}) = 1$. $\square$

Warning: $(xy)^{-1} \neq x^{-1}y^{-1}$ in general.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

Write the rows of the Cayley table as permutations of $\overset{1}{e}, \overset{2}{a}, \overset{3}{b}, \overset{4}{c}$:

$\{(\,), (12)(34), (13)(24), (14)(23)\}$ is a Klein four group as a subgroup of S_4.

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Gives $\{(\,), (1234), (13)(24), (1432)\}$ as a subgroup of S_4.

Theorem (Cayley Representation Theorem)
Every finite group G is isomorphic to a subgroup of S_n where $n = |G|$.