Algebra I

Group Theory

Book 3

A matrix in $GL_2(\mathbb{R})$ is conjugate to $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ iff it has trace 0 and determinant -1.

If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in GL_2(\mathbb{R})$ then A has characteristic polynomial $f(x)=\det(xI-A)=\det\left(\begin{bmatrix}x&0\\0&x\end{bmatrix}-\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)$

$$= \begin{vmatrix}x-a&-b\\-c&x-d\end{vmatrix} = (x-a)(x-d)-bc = x^2-\underbrace{(a+d)}_{\operatorname{tr}A}x+\underbrace{(ad-bc)}_{\det A}.$$

Some books define the characteristic polynomial of A as $\det(A-xI)=(-1)^n\underbrace{\det(xI-A)}_{\text{monic: its leading term is } x^n.}$

Cayley-Hamilton Theorem (look it up in any linear algebra book)

If $f(x)$ is the characteristic polynomial of an $n\times n$ matrix A, then $f(A)=0$.

In the 2×2 case, $A^2-(\operatorname{tr}A)A+(\det A)I=0$ holds as we compute here:

$$A^2=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix}$$

$$A^2-(\operatorname{tr}A)A+(\det A)I=\begin{bmatrix}a^2+bc&ab+bd\\ac+cd&bc+d^2\end{bmatrix}-(a+d)\begin{bmatrix}a&b\\c&d\end{bmatrix}+(ad-bc)\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}a^2+bc-(a+d)a+(ad-bc)&ab+bd-(a+d)b\\ac+cd-(a+d)c&bc+d^2-(a+d)d+(ad-bc)\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

If $A\in GL_2(\mathbb{R})$ has trace 0 and determinant -1 then it satisfies $A^2-\underline{0}A-1I=0$ so $A^2=I$

so in the group $GL_2(\mathbb{R})$, A has order ~~1 or~~ 2. ($\operatorname{tr} I=2$, not 0)

$f(x)=\det(xI-A)$ may or may not be the smallest degree polynomial that has A as a root.

The minimal polynomial of A, $m(x)$, is the monic polynomial of smallest degree satisfying $m(A)=0$.

Facts (see a linear algebra book):

- Roots of $f(x)$ are eigenvalues of A.
- $m(x)$ divides $f(x)$ i.e. $f(x)=h(x)m(x)$ for some monic polynomial $h(x)$ (often $h(x)=1$, $m(x)=f(x)$).
- Every eigenvalue of A is a root of $m(x)$.

Theorem Let $A \in GL_2(\mathbb{R})$. Then the following are equivalent:

(i) $\operatorname{tr} A = 0$, $\det A = -1$

(ii) A has order 2 but $A \neq -I$.

(iii) A is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We have proved (i) $\Rightarrow$ (ii). And (iii) $\Rightarrow$ (i) is easy. Assume $A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ for some $M \in GL_2(\mathbb{R})$.

Then $\operatorname{tr} A = \operatorname{tr}\left(M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}\right) = \operatorname{tr}\left(M^{-1}M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \operatorname{tr}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 0$.

$\operatorname{tr} AB = \operatorname{tr} BA$ if A is $m \times n$, B is $n \times m$ (short proof: see linear algebra. Both equal to $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} b_{ji}$)

$\det A = \det M \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\det M^{-1}}_{(\det M)^{-1}} = -1.$

$MM^{-1} = I$

$\det(M)\underbrace{\det(M^{-1})}_{1/\det M} = \det I = 1$

We must prove (ii) $\Rightarrow$ (iii). If A has order 2 then $A^2 = I$, $A \neq I$. A is a root of $x^2 - 1 = (x+1)(x-1)$

so the minimal poly. of A divides x^2-1: $m(x) = x^2-1$ or $x+1$ or $x-1$ or 1.

If $m(x) = 1$ then $m(A) = I = 0$. No!.

If $m(x) = x-1$ then $m(A) = A - I = 0$ then $A = I$ (No! I has order 1, not order 2)

If $m(x) = x+1$ then $m(A) = A + I = 0$ so $A = -I$ (No! by assumption).

So $m(x) = x^2 - 1$ divides $f(x)$, so $f(x) = x^2-1$. $\Rightarrow \operatorname{tr} A = 0$, $\det A = -1$. $\Rightarrow$ (i) holds

So ± 1 are eigenvalues of A. Let u, v be eigenvectors corresponding to $1, -1$ i.e. $Au = u$, $Av = -v$.

Let $M = [u|v]$ (2×2 matrix having u, v as columns)

$AM = [Au|Av] = [u|-v] = [u|v]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow A = M\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}M^{-1}$ i.e. (iii) holds. $\square$

There are two conjugacy classes of elements of order 2 in $G = GL_2(\mathbb{R})$:

- $\{-I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\}$ is in a class by itself since $-I \in Z(G)$
- All matrices conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ i.e. all matrices with trace 0 and determinant -1.
  This includes $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$, $a \in \mathbb{R}$