Algebra I

Group Theory

Book 2

Transpositions $(i\,j)$ are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd; and vice versa.
A cycle of odd length is an even permutation;
" " " even " " " odd " .

If α is a product of an even numbes of transpositions, then α is an even permutation.
" " " " " " " " odd " " " " " " " " " odd " .

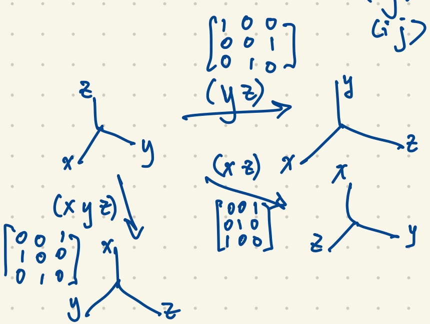
Permutations in S_5:

Even		Odd	
$()$	1	$(i\,j)$	10
$(i\,j\,k)$	20	$(i\,j\,k\,l)$	30
$(i\,j\,k\,l\,m)$	24	$(i\,j\,k)(l\,m)$	20
$(i\,j)(k\,l)$	15		
	60		60

$|S_5| = 120$

$A_5 = \{\text{even permutations in } S_5\}$

$|A_5| = 60$



An even permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-preserving transformation.

An odd permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-reversing transformation.

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then

$$\det T \begin{cases} = 0 & \text{if } T \text{ is not invertible} \\ > 0 & \text{" " preserves orientation} \\ < 0 & \text{" " " reverses " "} \end{cases}$$

A permutation $\alpha \in S_n$ can be expressed as a product of transpositions.

If α is a product of an even number of transpositions, then α is even.

" " " " " odd " " " " " " " odd.

In S_3:

$(1\,3)(1\,2)(1\,3)(2\,3)(2\,3)(2\,3)(1\,2)(2\,3) = (1\,2\,3)$ says $(1\,2\,3)$ is an even permutation.

$S_3 \cong \left\langle \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\-1&-1\end{bmatrix} \right\rangle \cong$ dihedral group of order 6 (symmetry group of an equilateral triangle)

n	no. of groups of order n up to isomorphism
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	5

Groups of order 2

$S_2 \cong \{0, 1\}$ mod 2 under addition $\cong \langle -1 \rangle$ under multiplication

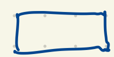
$\circ$	$()$	(12)
$()$	$()$	(12)
(12)	(12)	$()$

$+$	0	1
0	0	1
1	1	0

$\cdot$	1	-1
1	1	-1
-1	-1	1

Cayley tables of groups of order 2 all "look the same"

has a cyclic symmetry group of order 4



has an abelian symmetry group of order 4 which is not cyclic (the Klein four-group)

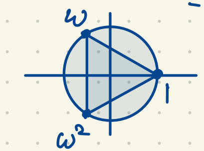
<u>Theorem</u> Any two groups of prime order are isomorphic; they are cyclic of order p.

Eg. $\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ (under addition mod 3) is isomorphic to $A_3 = \langle (123) \rangle = \{(), (123), (132)\}$ and $\{1, \omega, \omega^2\}$ under multiplication, $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∘	()	(123)	(132)
()	()	(123)	(132)
(123)	(123)	(132)	()
(132)	(132)	()	(123)

·	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω



We say two groups G, H are isomorphic ($G \cong H$) if there exists a bijection $\phi : G \to H$ such that $\phi(xy) = \phi(x)\phi(y)$

operation in G / operation in H

G

	y
x	xy

$\xrightarrow{\phi}$

H

	$\phi(y)$
$\phi(x)$	$\phi(xy)$

An isomorphism $\phi : \mathbb{Z}/3\mathbb{Z} \to A_3$ is a bijection satisfying $\phi(x+y) = \phi(x) \circ \phi(y)$

An isomorphism $\phi : \mathbb{R} \longrightarrow (0, \infty)$, $\phi(x+y) = \phi(x)\phi(y)$ is defined by $\phi(x) = e^x$

$\mathbb{R}$ under addition; $(0,\infty)$ under multiplication (subgroup of $\mathbb{R}^\times = (-\infty, 0) \cup (0, \infty)$)

$e^{x+y} = e^x \cdot e^y$.

$\ln = \phi^{-1} : (0, \infty) \longrightarrow \mathbb{R}$

$$\mathbb{R} \not\cong \mathbb{R}^\times$$

since $\mathbb{R}$ (reals under addition) has only one element of finite order whereas $\mathbb{R}^\times$ has two elements of finite order : ± 1.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}/3\mathbb{Z}$

is isomorphic to

*	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\phi(0) = c$
$\phi(1) = a$
$\phi(2) = b$

*	c	a	b
c	c	a	b
a	a	b	c
b	b	c	a

or $\phi(0) = c$
$\phi(1) = b$
$\phi(2) = a$

*	c	b	a
c	c	b	a
b	b	a	c
a	a	c	b

+	0
0	0

Every group of order 1 is isomorphic to

·	1
1	1

(trivial group $\{1\}$)

" " " " " " 2 " " " " $\mathbb{Z}/2\mathbb{Z}$

+	0	1
0	0	1
1	1	0

	c
a	ac
b	bc

If $ac = bc$ then multiply both sides by c^{-1} on the right to get

$$(ac)c^{-1} = (bc)c^{-1}$$
$$a(cc^{-1}) = b(cc^{-1})$$
$$a1 = b1$$
$$a = b$$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Every group of order 3 is cyclic (isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under addition).