Math 3500

Algebra I: Group Theory

Book 2

Similar to HW#2: How many elements of each order does S_4 have?

1 element of order 1 : $()$ = identity,

9 elements of order 2 : (12), (13), (14), (23), (24), (34), $\leftarrow$ $\binom{4}{2} = 6$ transpositions

$(12)(34)$, $(13)(24)$, $(14)(23)$ $\leftarrow$ $\frac{1}{2!}\binom{4}{2}\binom{2}{2} = \frac{6}{2} = 3$

8 elements of order 3: (123), (124), (132), (134), (142), (143), (234), (243)

6 elements of order 4: (1234), (1243), (1324), (1342), (1423), (1432)

$24 = 4! = |S_4|$ $\qquad |S_n| = n! = 1 \times 2 \times 3 \times \cdots \times n$

In S_n the number of n-cycles is $(n-1)!$

The binomial coefficient $\binom{n}{k}$ ("n choose k") is the number of ways to choose a subset of size k from a set of size n.

By the way, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$\binom{n}{k}$ = k^{th} entry in n^{th} row of Pascal's Triangle

Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\binom{4}{2}$ = number of 2-subsets in $[4] = \{1,2,3,4\}$
$= 6$

Pascal's Triangle

k=0, k=1, k=2, k=3, k=4, k=5, k=6, k=7

n=0: 1
n=1: 1 1
n=2: 1 2 1
n=3: 1 3 3 1
n=4: 1 4 6 4 1
n=5: 1 5 10 10 5 1
n=6: 1 6 15 20 15 6 1
n=7: 1 7 21 35 35 21 7 1

A ~~transposition~~ is a 2-cycle $(i\ j) \in S_n$, $i \neq j$ in $[n] = \{1, 2, \cdots, n\}$.

Products of disjoint transpositions e.g. $(1\ 3)(2\ 5)(6\ 8) \in S_8$ are elements of order 2.

How many elements of order 2 are there in S_7?

Transpositions: $(12), (13), (14), \cdots, (67)$ i.e. $(i\ j)$ where $i \neq j$ in $[7] = \{1, 2, \cdots, 7\}$

$\binom{7}{2} = 21$ transpositions

Products of two disjoint transpositions e.g. $(2\,6)(3\,4) = (3\,4)(2\,6)$

Number of these is $\frac{1}{2}\underbrace{\binom{7}{2}}_{21}\underbrace{\binom{5}{2}}_{10} = 105$.

Products of three disjoint transpositions eg. $(1\,5)(2\,7)(3\,6) = (15)(36)(27) = (27)(15)(36) = \cdots$

Number of these is $\underbrace{\frac{1}{3!}}_{\frac{1}{6}}\underbrace{\binom{7}{2}}_{21}\underbrace{\binom{5}{2}}_{10}\underbrace{\binom{3}{2}}_{3} = \frac{21 \times 10 \times \cancel{3}}{\cancel{6}\ 2} = 105$

Number of 3-cycles in S_7 eg. $(2\,7\,4)$: $2\binom{7}{3} = 2 \times 35 = 70$

Number of products of two disjoint 3-cycles: eg. $(2\,7\,4)(3\,5\,6) = (3\,5\,6)(2\,7\,4)$

$\cancel{\frac{1}{2}} \cdot 70 \cdot \binom{4}{3} \cdot \cancel{2} = 70 \cdot 4 = 280$

$280 + 70 = 350$ elements of order 3 in S_7

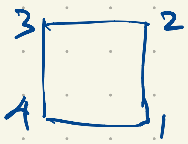
Elements of order 12 in S_7 eg. $(1\,4\,2)(3\,7\,5\,6)$

$70 \cdot 3! = 70 \cdot 6 = 420$ elements of order 12 in S_7.

Revisiting the dihedral group of order 8 (symmetry group of a square)

$G = \{ I, R, R^2, R^3, D, D', V, H \}$

viewed as a group of permutations of the four vertices



$I \longmapsto ()$

$R \longmapsto (1234)$

$R^2 \longmapsto (13)(24)$

$R^3 \longmapsto (1432)$

$H \longmapsto (12)(34)$

$V \longmapsto (14)(23)$

$D \longmapsto (13)$

$D' \longmapsto (24)$

$G \cong$ subgroup of S_4: $\{ (), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24) \}$

$= \langle (1234), (13) \rangle$

$RV = (1234)(14)(23) = (24) = D'$

This is an example of a <u>permutation group</u> of degree 4, i.e. a subgroup of S_4.

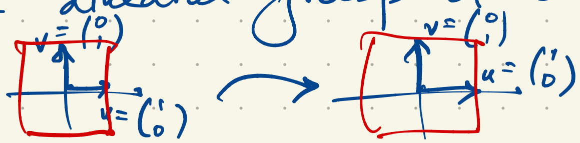
A <u>permutation group</u> of degree n is a subgroup of the symmetric group S_n.

<u>Theorem</u> (Cayley's Representation Theorem) Every finite group is isomorphic to a permutation group. In fact, if $|G| = n$ then G is isomorphic to a subgroup of S_n. (But we can usually do better. Eg. the dihedral group of order 8 is isomorphic to a subgroup of S_8. But S_4 is even better.)

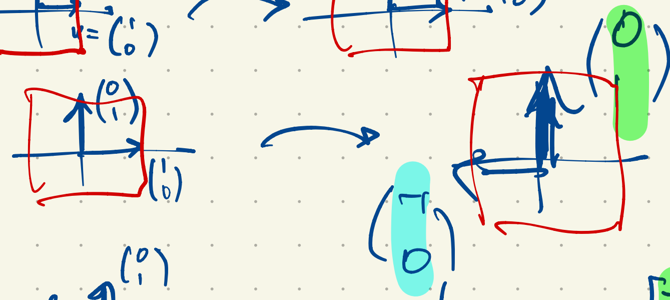
The two most important general classes of examples of groups are (i) <u>permutation groups</u> (i.e. subgroups of S_n) and

(ii) <u>linear groups</u> (i.e. subgroups of $GL_n(F) = \{$invertible $n \times n$ matrices over a field $F\}$). eg. $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$ (integers mod p)

The dihedral group of order 8 is also a subgroup of $GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0 \right\}$.

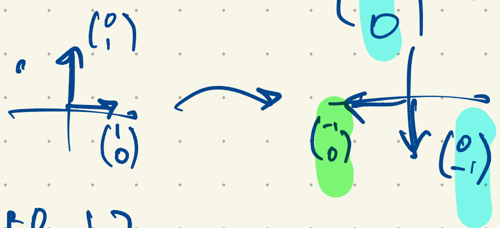
I:  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

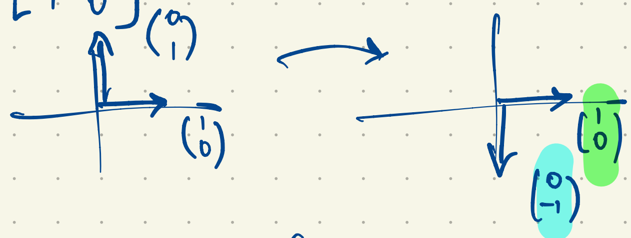
R:  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

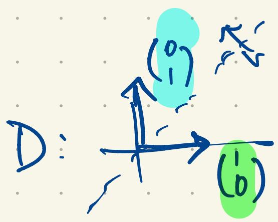
(the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ rotated 90° counter-clockwise about the origin)

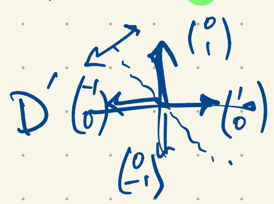
R^2:  $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$R^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

H:  $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$G = \text{symmetry group of square} \cong \left\{ \underset{}{\overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}, \overset{R}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}, \overset{R^2}{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}, \overset{R^3}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \overset{H}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \underset{V}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}, \underset{D}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}, \underset{D'}{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}} \right\}$$

D:  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

D': 

subgroup of $GL_2(\mathbb{R})$

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$$

$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ commutes with every element of G (not so immediately obvious from the other ways of representing G).

Similar to HW2 #4, 5 : $\mathbb{F}_p = \{0, 1, \cdots, p-1\}$ integers mod p.

eg. $p=3$, $F = \mathbb{F}_3 = \{0,1,2\}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\frac{1}{2} = 2$

$-\frac{1}{2} = \quad -\frac{1}{2} = -2 = 1$

$F^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in F \right\}$

$|F^2| = 3^2 = 9$

2×2 matrices over F there are $\underline{3^4 = 81}$ 2×2 matrices over F. $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$

$GL_2(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in F, \underbrace{ad-bc \neq 0}_{} \right\}$

i.e. $ad-bc = 1$ or 2 i.e. $ad-bc = \pm 1$.

$|GL_2(F)| = 8 \times 6 = 48$

$9-1=8$ choices for first column to be nonzero

$9-3=6$ choices for the second column to be not a scalar multiple of the first column.

eg. $g = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in GL_2(F) \Rightarrow g^2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

so $|g| = 2$.

Eg. $F = \mathbb{F}_2 = \{0,1\}$

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

i.e. $\det A = 1$

Let's try $G = GL_3(F) = \{3\times 3 \text{ matrices } A \text{ over } F \text{ such that } \det A \neq 0\}$

The total number of 3×3 matrices over F is $2^9 = 512$.

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$|G| = 7 \times 6 \times 4 = 168$

$8-1$ nonzero vectors

$8-2$

$8-4$

F^3 is a 3-dimensional vector space over F

$|F^3| = 2^3 = 8$

eg. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in G$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{A^2}\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^4 = A^2A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$|A| = 4 =$ the order of A

Note: $4 \mid 168$

$\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$

$\mathbb{F}_7^\times = \{1,2,3,4,5,6\}$ multiplicative group of order 6

$\langle 1 \rangle = \{1\}$

$\langle 2 \rangle = \{1,2,4\}$

$\langle 3 \rangle = \{1,3,2,6,4,5\} = \langle 5 \rangle$

$\langle 4 \rangle = \{1,4,2\} = \langle 2 \rangle$

$\langle 5 \rangle = \{1,5,4,6,2,3\}$

$\langle 6 \rangle = \{1,6\}$

x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

=

x	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2	6	4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

$\mathbb{Z}/6\mathbb{Z} = \{0,1,2,3,4,5\}$ under addition mod 6

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

We have an isomorphism from $(\mathbb{Z}/6\mathbb{Z}, +)$ to $(\mathbb{F}_7^\times, \times)$

defined by $\phi : \mathbb{Z}/6\mathbb{Z} \longrightarrow \mathbb{F}_7^\times$

$\phi(0) = 1$

$\phi(1) = 3$

$\phi(2) = 2$

$\phi(3) = 6$

$\phi(4) = 4$

$\phi(5) = 5$

$\phi(x+y) = \phi(x)\phi(y)$

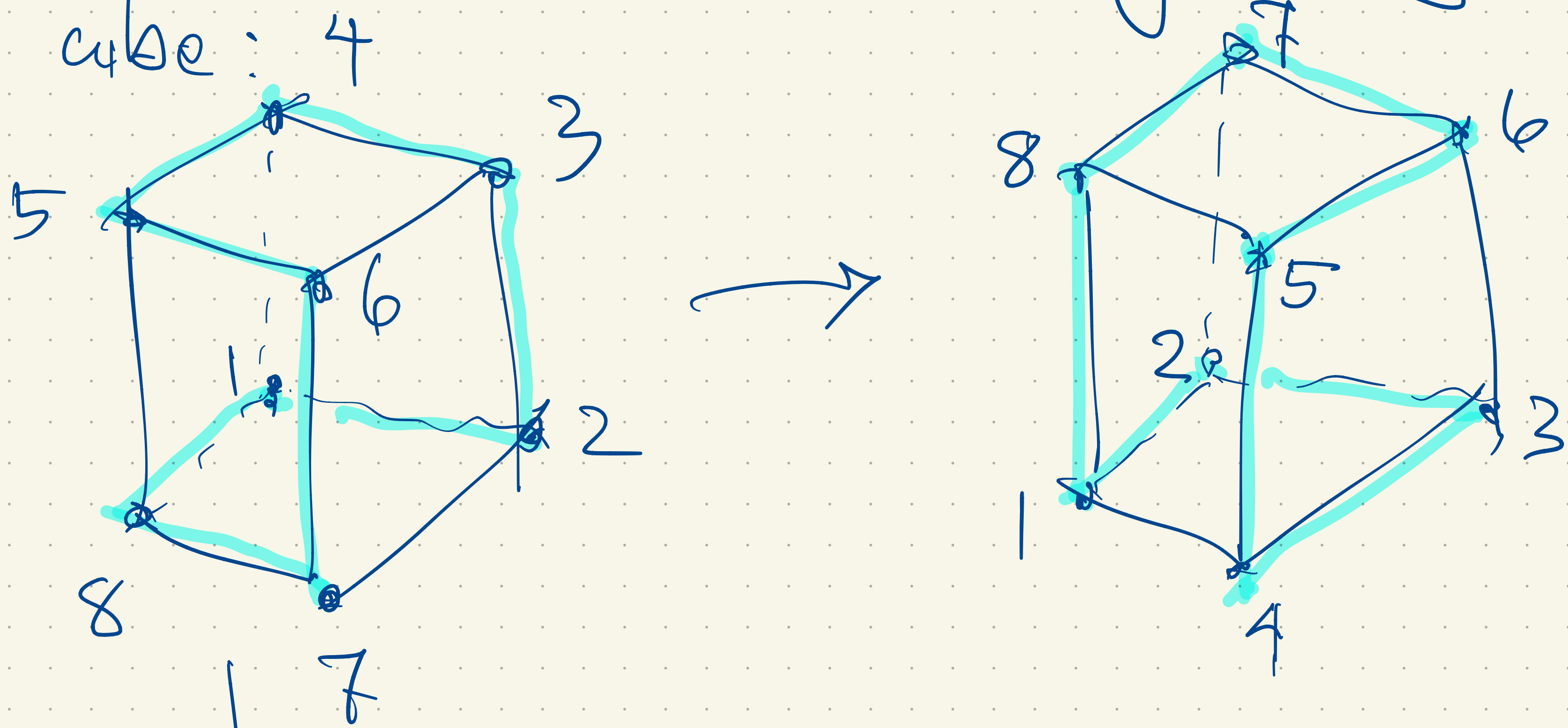
$\phi(\underbrace{1+1+\cdots+1}_{k \text{ times}}) = \underbrace{3 \cdot 3 \cdots 3}_{k\text{ times}}$

$\phi(k) = 3^k$

$\phi(k+l) = 3^{k+l} = 3^k \cdot 3^l$

$= \phi(k)\phi(l)$

check that (123658)(47) is a symmetry of my cube:



(47) is not a symmetry