Algebra I

Group Theory

Book 2

Transpositions $(i\,j)$ are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd; and vice versa.
A cycle of odd length is an even permutation;
" " " even " " " odd " .

If α is a product of an even number of transpositions, then α is an even permutation.
" " " " " " odd " " " " " " " odd " .

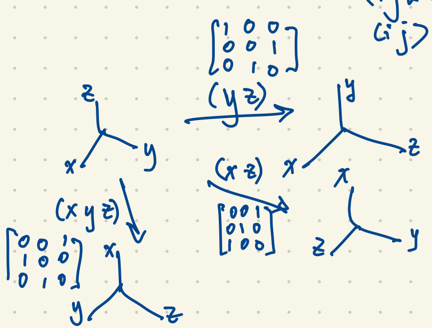
Permutations in S_5:

Even		Odd	
$()$	1	$(i\,j)$	10
$(i\,j\,k)$	20	$(i\,j\,k\,l)$	30
$(i\,j\,k\,l\,m)$	24	$(i\,j\,k)(l\,m)$	20
$(i\,j)(k\,l)$	15		
	60		60

$|S_5| = 120$

$A_5 = \{\text{even permutations in } S_5\}$

$|A_5| = 60$



An even permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-preserving transformation.

An odd permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-reversing transformation.

If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then

$$\det T \begin{cases} = 0 & \text{if } T \text{ is not invertible} \\ > 0 & \text{" " preserves orientation} \\ < 0 & \text{" " reverses "} \end{cases}$$

A permutation $\alpha \in S_n$ can be expressed as a product of transpositions.
If α is a product of an even number of transpositions, then α is even.
" " " " " " odd " " " " " " " " odd.

In S_3:
$(13)(12)(13)(23)(23)(23)(12)(23) = (1\ 2\ 3)$ says (123) is an even permutation.

$S_3 \cong \left\langle \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\-1&-1\end{bmatrix} \right\rangle \cong$ dihedral group of order 6 (symmetry group of an equilateral triangle)

n	no. of groups of order n up to isomorphism
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	5

Groups of order 2

$S_2 \cong \{0, 1\}$ mod 2 under addition $\cong \langle -1 \rangle$ under multiplication

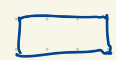
$\circ$	$()$	(12)
$()$	$()$	(12)
(12)	(12)	$()$

$+$	0	1
0	0	1
1	1	0

$\cdot$	1	-1
1	1	-1
-1	-1	1

Cayley tables of groups of order 2 all "look the same"


has a cyclic symmetry group of order 4


has an abelian symmetry group of order 4 which is not cyclic (the Klein four-group)

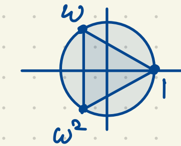
Theorem Any two groups of prime order are isomorphic; they are cyclic of order p.

Eg. $\mathbb{Z}/3\mathbb{Z} = \{0,1,2\}$ (under addition mod 3) is isomorphic to $A_3 = \langle(123)\rangle = \{(), (123), (132)\}$ and $\{1, \omega, \omega^2\}$ under multiplication, $\omega = \frac{-1+i\sqrt{3}}{2} = e^{2\pi i/3}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

∘	()	(123)	(132)
()	()	(123)	(132)
(123)	(123)	(132)	()
(132)	(132)	()	(123)

·	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω



We say two groups G, H are <u>isomorphic</u> ($G \cong H$) if there exists a bijection $\phi: G \to H$ such that $\phi(xy) = \phi(x)\phi(y)$

operation in G (for xy) — operation in H (for $\phi(x)\phi(y)$)

G:

		y
x		xy

$\xrightarrow{\phi}$ H:

		$\phi(y)$
$\phi(x)$		$\phi(xy)$

An isomorphism $\phi: \mathbb{Z}/3\mathbb{Z} \to A_3$ is a bijection satisfying $\phi(x+y) = \phi(x)\circ\phi(y)$

An isomorphism $\phi: \mathbb{R} \longrightarrow (0,\infty)$, $\phi(x+y) = \phi(x)\phi(y)$ is defined by $\phi(x) = e^x$

$\mathbb{R}$ under addition; $(0,\infty)$ under multiplication (subgroup of $\mathbb{R}^\times = (-\infty,0)\cup(0,\infty)$)

$e^{x+y} = e^x \cdot e^y$.

$\ln = \phi^{-1}: (0,\infty) \longrightarrow \mathbb{R}$

$\mathbb{R} \ncong \mathbb{R}^\times$

since $\mathbb{R}$ (reals under addition) has only one element of finite order whereas $\mathbb{R}^\times$ has two elements of finite order: ± 1.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}/3\mathbb{Z}$

is isomorphic to

*	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$\phi(0) = c$
$\phi(1) = a$
$\phi(2) = b$

*	c	a	b
c	c	a	b
a	a	b	c
b	b	c	a

or $\phi(0) = c$
$\phi(1) = b$
$\phi(2) = a$

*	c	b	a
c	c	b	a
b	b	a	c
a	a	c	b

+	0
0	0

Every group of order 1 is isomorphic to

·	1
1	1

(trivial group $\{1\}$)

" " " " 2 " " $\mathbb{Z}/2\mathbb{Z}$

+	0	1
0	0	1
1	1	0

	c
a	ac
b	bc

If $ac = bc$ then multiply both sides by c^{-1} on the right to get

$$(ac)c^{-1} = (bc)c^{-1}$$
$$a(cc^{-1}) = b(cc^{-1})$$
$$a1 = b1$$
$$a = b$$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Every group of order 3 is cyclic (isomorphic to $\mathbb{Z}/3\mathbb{Z}$ under addition).

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Two cases: either all non-identity elements of G have order 2, or G has an element not of order 2.

Theorem: There are exactly two groups of order 4 up to isomorphism: the Klein four-group and the cyclic group of order 4.

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

cyclic group of order 5

$$\langle a\rangle = \{e, a, \underset{b}{a^2}, \underset{c}{a^3}, \underset{d}{a^4}\}$$

	e	a	b	c	d
e	e	a	b	c	d
a	a	e	c	d	b
b	b	c	d	a	e
c	c	d	e	b	a
d	d	b	a	e	c

is not a group!

It is a quasigroup, in fact since it has an identity e, it is a loop (its Cayley table is a Latin square: each row/column is a permutation of e,a,b,c,d).

This loop is not associative

eg. $(ca)d = dd = c$

$c(ad) = cb = e$

c is a left inverse ~~for~~ b $(cb=e)$ but not a right inverse ~~for~~ b $(bc=a)$.

Theorem If every non identity element of a group G has order 2, then G is abelian.

Proof (Note: $x^2=e=$ identity for every $x\in G$.)

Let $x,y\in G$. Then $(xy)^2 = xyxy = e$ so

$$yx = \underbrace{\cancel{x}(x}_{x^2=e}yxy\underbrace{)\cancel{y}}_{y^2=e} = xey = xy.$$

$\square$

In such groups, $x^{-1}=x$ for all $x\in G$.

Shoe-Sock Theorem

In every group G, with identity 1, for $x, y \in G$ we have $(xy)^{-1} = y^{-1}x^{-1}$.

Proof $(y^{-1}x^{-1})(xy) = y^{-1}1y = 1$ and $(xy)(y^{-1}x^{-1}) = 1$. $\square$

Warning: $(xy)^{-1} \neq x^{-1}y^{-1}$ in general.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein four-group

Write the rows of the Cayley table as permutations of $\overset{1}{e}, \overset{2}{a}, \overset{3}{b}, \overset{4}{c}$:

$\{(), (12)(34), (13)(24), (14)(23)\}$ is a Klein four group as a subgroup of S_4.

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Cyclic group of order 4

Gives $\{(), (1234), (13)(24), (1432)\}$ as a subgroup of S_4.

Theorem (Cayley Representation Theorem)
Every finite group G is isomorphic to a subgroup of S_n where $n = |G|$.

By the way, every finite group G is also isomorphic to a group of matrices under multiplication.

Theorem If G is a finite group of order n, then every element $g\in G$ has order dividing n.
(If $g\in G$ then $|g| \mid n$.)

Eg. S_4 has elements of order $1,2,3,4$. These orders of elements divide $|S_4| = 24$.
S_5 has elements of order $1,2,3,4,5,6$ (divisors of $|S_5| = 120$).

Proof In the general case this follows from a later theorem, Lagrange's Theorem.
Here let's prove the theorem in the special case that G is abelian. (We have already proved the result for cyclic groups.)
Consider the product of all the group elements $\pi = g_1g_2g_3\cdots g_n$ where $G=\{g_1,g_2,\dots,g_n\}$, $g_1=1$.
Note: since G is abelian, π is well-defined; it doesn't depend on what order we list the elements $g_1,\dots,g_n\in G$. Pick $a\in G$. (So $a\in\{g_1,\dots,g_n\}$.) The elements $ag_1, ag_2, \dots, ag_n$ are again all the elements of G so

$$\begin{array}{c|cccc} & g_1 & g_2 & \cdots & g_n \\ \hline a & ag_1 & ag_2 & ag_3\cdots & ag_n \end{array}$$

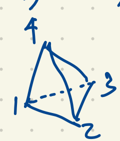
$$(ag_1)(ag_2)(ag_3)\cdots(ag_n) = \pi = a^n g_1g_2\cdots g_n = a^n\pi$$

So $a^n=1$ and $k=|a|$ must divide n. $\square$

Lagrange's Theorem If G is any finite group of order n, and $H\le G$ (i.e. H is a subgroup of G) then $|H| \mid n$.

This generalizes the previous statement: if $g\in G$ then by Lagrange's Theorem, $|g| = |\langle g\rangle| \,\big|\, |G|$.

eg. $|A_4| = \frac{1}{2}|S_4| = 12$, $A_4 = \{(), (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$.

The symmetry group of a regular tetrahedron 

is isomorphic to S_4.

The rotational symmetry group of the regular tetrahedron (the direct isometry group, consisting of those symmetries that preserve orientation) is isomorphic to A_4.

$A_4 = \{(),\ (123),\ (124),\ (132),\ (134),\ (142),\ (143),\ (234),\ (243),\ (12)(34),\ (13)(24),\ (14)(23)\}$.

Subgroups of A_4 have order 1, 2, 3, 4.

Elements of A_4 have order 1, 2, 3.

Divisors of $|A_4| = 12$ are 1, 2, 3, 4, 6, 12.

$\langle (243), (12)(34) \rangle = \{(),\ (243),\ (12)(34),\ (234),\ (142),\ (124),\ \cdots\} = A_4$.

$(243)(12)(34) = (142)$

$\{(),\ (12)(34),\ (13)(24),\ (14)(23)\}$ is the Klein four-group, a subgroup of A_4.

Question: How many subgroups of $\mathbb{Z}$ are there containing 4? (Note: $\mathbb{Z}$ is an additive group.)

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots\}$

$2\mathbb{Z} = \{\ldots, -6, -4, -2, 0, 2, 4, 6, 8, \ldots\}$

$4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, 12, \cdots\}$

$-4\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, 12, \cdots\}$

Answer: There are three subgroups of $\mathbb{Z}$ containing 4, namely $\mathbb{Z}$, $2\mathbb{Z}$, $4\mathbb{Z}$.

$\mathbb{Z}$ has infinitely subgroups: one finite subgroup $\{0\}$ and all the other subgroups are infinite.

There are infinite subgroups of $\mathbb{Z}$ containing 4 but not infinitely many subgroups of $\mathbb{Z}$ containing 4.

Note: For every cyclic group G, all subgroups of G are cyclic; they are generated by powers of the generator of G.

Eg. $G = \langle g \rangle$ where $|g| = \infty$ i.e. $|G| = |\langle g \rangle| = |g| = \infty$.

$= \{\ldots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \ldots\}$ with no repeats.

1 is the identity

$g^i g^j = g^{i+j} = g^j g^i$

How many subgroups of $G = \langle g \rangle$ contain g^4? Three: $\langle g \rangle, \langle g^2 \rangle, \langle g^4 \rangle$.

$\langle g \rangle = \langle g^{-1} \rangle$, $\langle g^2 \rangle = \langle g^{-2} \rangle = \langle g^6, g^{10} \rangle$, $\langle g^4 \rangle = \langle g^{-4} \rangle$

$G = \{\ldots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, g^4, \ldots\}$

$\langle g^2 \rangle = \{\ldots, g^{-6}, g^{-4}, g^{-2}, 1, g^2, g^4, g^6, \ldots\}$

$\langle g^4 \rangle = \{\ldots, g^{-8}, g^{-4}, 1, g^4, g^8, g^{12}, \ldots\}$

$\langle g^6, g^{10} \rangle \le \langle g^2 \rangle$

$\langle g^2 \rangle \le \langle g^6, g^{10} \rangle$

since $g^2 = (g^6)^2 (g^{10})^{-1}$

So $\langle g^2 \rangle = \langle g^6, g^{10} \rangle$

$\underbrace{G}_{\text{multiplicative cyclic group}} \cong \underbrace{\mathbb{Z}}_{\text{additive cyclic group}}$

$\phi : \mathbb{Z} \to G$ is an isomorphism

$\phi(i) = g^i$

Theorem If G is a group of even order, then G has an element of order 2 (i.e. at least one element of order 2). Note: G is not necessarily abelian.

Proof Pair up each group element with its inverse giving pairs $\{g, g^{-1}\}$ for $g \in G$.

Note that $g = g^{-1}$ iff g has order 1 or 2. ($g = g^{-1} \Leftrightarrow g^2 = 1 \Leftrightarrow |g|$ divides 2). So G is partitioned into subsets $\{g, g^{-1}\}$ having size 1 or 2. If G has no elements of order 2 then we have partitioned a set G of even cardinality into one subset $\{1\}$ of size 1, and a collection of pairs $\{g, g^{-1}\}$ of size 2, a contradiction. $\square$

what we actually showed is that in a group of even order, the number of elements of order 2 is odd. (In a group of odd order, there are no elements of order 2 although we haven't proved this yet except in the abelian case.)

Eg. Direct Products: Given groups G, H (say, multiplicative) we form the <u>direct product</u> of G and H as $G \times H = \{(g,h) : g \in G,\ h \in H\}$ (the cartesian product of the sets G and H) which becomes a group under <u>coordinatewise</u> <u>multiplication</u> i.e.

$$(g,h)(g',h') = (gg', hh')$$

and coordinatewise inverses i.e. $(g,h)^{-1} = (g^{-1}, h^{-1})$
and the coordinatewise identity $1 \in G \times H$ is $1 = 1_{G\times H} = (1_G, 1_H)$. or $e_{G\times H} = (e_G, e_H)$.

Eg. $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ under addition mod 2

+	0	1
0	0	1
1	1	0

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(x,y) : x,y \in \mathbb{Z}/2\mathbb{Z}\} = \{(0,0), (0,1), (1,0), (1,1)\}$$

$(x,y) + (x',y') = (x+x', y+y')$. The identity $0 = (0,0)$.

This is the Klein four-group since it has 3 elements of order 2.

Note: Many books write $\mathbb{Z}_2$ or Z_2 in place of $\mathbb{Z}/2\mathbb{Z}$

If $|G| = m$ and $|H| = n$ then $|G \times H| = mn$.

If G and H are abelian then so is $G\times H$.
In fact, the converse holds: G and H are both abelian, iff $G\times H$ is abelian.

$G \times H \cong H \times G$
$\phi: G\times H \to H \times G$
$\phi(g,h) = (h,g)$ is an isomorphism.

$G \times H$ has a subgroup $G \times \{1_H\} = \{(g, 1_H) : g \in G\} \cong G$

An isomorphism $G \times \{1_H\} \longrightarrow G$ is given by $(g, 1_H) \longmapsto g$.

Likewise, $G \times H$ has a subgroup $\{1_G\} \times H \cong H$

$$\underbrace{(g, 1_H)}_{\substack{\uparrow \\ G \times \{1_H\} \\ \cong \\ G}} \underbrace{(1_G, h)}_{\substack{\uparrow \\ \{1_G\} \times H \\ \cong \\ H}} = (g, h) = (1_G, h)(g, 1_H)$$

Eg. $\underbrace{\mathbb{R}^{\times} = (-\infty, 0) \cup (0, \infty)}_{\text{multiplicative group}} \cong \underbrace{\mathbb{R}}_{\text{additive group}} \times \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{additive}}$

An isomorphism $\phi : \mathbb{R}^{\times} \longrightarrow \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ is $\phi(a) = \begin{cases} (\ln|a|, 0) & \text{if } a > 0 \\ (\ln|a|, 1) & \text{if } a < 0 \end{cases}$

It's easy to see that ϕ is one-to-one and onto.

We show that $\phi(ab) = \phi(a) + \phi(b)$ for all $a, b \in \mathbb{R}^{\times}$.

We argue in ~~four~~ cases. If $a, b > 0$ then

$\phi(ab) = (\ln|ab|, 0)$ since $ab > 0$

$= (\ln|a| + \ln|b|, 0) = (\ln|a|, 0) + (\ln|b|, 0) = \phi(a) + \phi(b)$

If $a, b < 0$ then $ab > 0$ so

$\phi(ab) = (\ln|ab|, 0) = (\ln|a|, 1) + (\ln|b|, 1)$

$= \phi(a) + \phi(b)$

If $a > 0 > b$ then $ab < 0$ so

$\phi(ab) = (\ln|ab|, 1) = (\ln|a|, 0) + (\ln|b|, 1) = \phi(a) + \phi(b)$

Similarly if $a < 0 < b$.

Every cyclic group is abelian.
Not every abelian group is cyclic but every abelian group is a direct product of cyclic groups.
eg. the Klein four-group is a direct product of two groups of order 2 i.e. $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

There are five groups of order 8 up to isomorphism:

$\left.\begin{array}{l} \mathbb{Z}/8\mathbb{Z} \quad \text{(cyclic)} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} = \{(a,b) : a \in \mathbb{Z}/2\mathbb{Z},\ b \in \mathbb{Z}/4\mathbb{Z}\}. \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{(a,b,c) : a,b,c \in \mathbb{Z}/2\mathbb{Z}\} \text{ under addition} \end{array}\right\}$ three abelian groups of order 8

dihedral group of order 8 $\cong$ symmetry group of square, D_4 (sometimes D_8)
quaternion group of order 8, Q or Q_8

$Q = \{1, \underset{\text{order } 2}{-1}, \underbrace{i, -i, j, -j, k, -k}_{\text{order } 4}\}$

$ij = k,\ ji = -k,\quad i^2 = j^2 = k^2 = -1$
$jk = i,\ kj = -i$
$ki = j,\ ik = -j$

For any field F (eg. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$) $GL_n(F) = \{\text{invertible } n\times n \text{ matrices over } F\}$ i.e. having entries in F.
Also $F = \mathbb{F}_3 = \{0,1,2\}$ works with addition mod 3. $2+2 = 1 = 2\times 2$
$\frac{1}{2} = 2$

In $\mathbb{F}_7 = \{0,1,2,\dots,6\}$, $\frac{1}{5} = 3$.
$\mathbb{F}_p = \{0,1,2,\dots,p-1\}$ is a field whenever p is prime.
$GL_2(\mathbb{F}_3) = \{\text{invertible } 2\times 2 \text{ matrices over } \mathbb{F}_3\}$ is a group of order 48.
$GL_2(\mathbb{R}) = \{\text{invertible } 2\times 2 \text{ matrices over } \mathbb{R}\} = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R},\ ad-bc \neq 0\right\}$
$GL_n(F) = \{\text{invertible } n\times n \text{ matrices over } F\}$ = general linear group of degree n over F
also denoted $GL(n,F)$ in the textbook

$SL_n(F)$ is the special linear group of degree n over F ; $SL_n(F) \le GL_n(F)$
or $SL(n,F)$ $SL_n(F) = \{n\times n$ matrices over F having determinant $1\}$.

If $F = \mathbb{F}_p = \{0,1,2,\cdots,p-1\} \bmod p$ (field of prime order p) then we can count elements in $GL_n(\mathbb{F}_p)$ or $SL_n(\mathbb{F}_p)$.

(For 2×2 matrix over $\mathbb{F}_3$, $\underline{33}$ matrices have $\det A = 0$, $\underline{24}$ matrices have $\det A = 1$, $\underline{24}$ ... " $\det A = 2$).

$|GL_2(\mathbb{F}_3)| = 48$.

The number of 2×2 matrices over $\mathbb{F}_3 = \{0,1,2\}$ is 81. How many of them are invertible?
We count invertible matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a,b,c,d \in F = \mathbb{F}_3$ with linearly independent columns.
There are $\underline{\ 8\ }$ choices for the first column $\begin{bmatrix} a \\ c \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $9-1=8$
Having chosen the first column $\begin{bmatrix} a \\ c \end{bmatrix}$, there are $\underline{\ 6\ }$ choices for the second column $\begin{bmatrix} b \\ d \end{bmatrix}$ which are not a scalar multiple of the first column. $9-3=6$ So $|GL_2(\mathbb{F}_3)| = 8\times 6 = 48$.
In fact, for $A \in GL_2(F)$, $F = \mathbb{F}_3$, there are 24 choices with determinant 1, and 24 choices with determinant $-1 = 2$.

$$|GL_n(\mathbb{F}_p)| = \underbrace{(p^n-1)}_{\text{no. of choices of first column}}\underbrace{(p^n-p)}_{\text{no. of choices of second column}}\underbrace{(p^n-p^2)}_{\text{no. of choices of third column}}\cdots\underbrace{(p^n-p^{n-1})}_{\text{no. of choices of last column}}$$

$|GL_2(\mathbb{F}_p)| = (p^2-1)(p^2-p)$
For $A \in GL_n(\mathbb{F}_p)$, $\det A \in \{1,2,\cdots,p-1\}$ and there equally many matrices with each possible nonzero determinant in $\{1,2,\cdots,p-1\}$ so

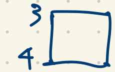
$$|SL_n(\mathbb{F}_p)| = \frac{1}{p-1}|GL_n(\mathbb{F}_p)|.$$

We'll explain later.

For any group G, the center of G is $Z(G) = \{$all elements in G which commute with everything in $G\}$

Zentrum ← not $\mathbb{Z}$ Zahlen $\qquad = \{z \in G : zx = xz \text{ for all } x \in G\}$

Eg. if G is the symmetry group of a square (a dihedral group of order 8) then $|Z(G)| = 2$ and $Z(G)$ consists of the identity and the half-turn (180° rotation about the center).



If we represent G using permutations on the vertices 1,2,3,4 then

$$G = \{(),\ (1234),\ (13)(24),\ (1432),\ (12)(34),\ (14)(23),\ (13),\ (24)\}$$

then $Z(G) = \langle (13)(24) \rangle = \{(),\ (13)(24)\}$.

Alternatively, G can be represented as a subgroup of $GL_2(\mathbb{R})$:

$$G = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&-1\\1&0\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix}, \begin{bmatrix}0&1\\-1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}-1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&-1\\-1&0\end{bmatrix} \right\}$$

$$Z(G) = \left\langle \begin{bmatrix}-1&0\\0&-1\end{bmatrix} \right\rangle = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}-1&0\\0&-1\end{bmatrix} \right\rangle$$

In general, $Z(G) \leq G$ (a subgroup of G).

$Z(G) = G$ iff G is abelian.

For many groups, $Z(G) = \{1\}$ ← identity $\qquad$ eg. $Z(S_3) = \{()\}$.

e = identity of G

<u>Theorem</u> If G is a group and $z \in G$, then $Z(G) \leq G$ (the center of G is a subgroup of G).

<u>Proof</u> Since $eg = g = ge$ for every $g \in G$, $e \in Z(G)$. If $z, z' \in Z(G)$ then

$$(zz')g = z(z'g) = z(gz') = (zg)z' = (gz)z' = g(zz')$$

so $zz' \in Z(G)$. Also if $z \in Z(G)$ then for every $g \in G$ we have $zg = gz$ so $z^{-1}g = z^{-1}(gz)z^{-1} = z^{-1}(zg)z^{-1} = gz^{-1}$ so $z^{-1} \in Z(G)$. $\square$

Let $S \subseteq G$. The <u>centralizer</u> of S in G is $C_G(S)$ = the set all all elements of G commuting with every element of S, i.e. $C_G(S) = \{g \in G : gs = sg \text{ for all } s \in S\}$.

eg. $C_G(e) = G$, $C_G(G) = Z(G)$. If $z \in Z(G)$ then $C_G(z) = G$.

In S_4, $C_{S_4}((12)) = \{(),\ (34),\ (12),\ (12)(34)\}$

In general, $C_G(S) \leq G$ (the centralizer of a subset of G is always a subgroup of G).
The proof of this is virtually identical to the proof above; just quantify over $g \in S$ rather than $g \in G$.

If $G = GL_n(F)$ = invertible $n \times n$ matrices over F, then $Z(G) = \{\lambda I : \lambda \neq 0 \text{ in } F\}$

$I = I_n$ = $n \times n$ identity matrix.

$$\left.\begin{array}{l} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \end{array}\right\} \text{ so } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \notin Z(GL_2(\mathbb{R}))$$

Let $E_{ij}(a) = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} 1 & & a \\ & \ddots & \\ & & 1 \end{bmatrix} \end{array}$ for $i \neq j$. (This is the elementary matrix obtained from the identity matrix by adding an "a" in the (i,j) position.)

If $A = [a_{ij} : 1 \leq i, j \leq n] \in Z(GL_n(F))$ then $A E_{ij}(1) = E_{ij}(1) A$ so $a_{ij} = 0$. So A is diagonal.
Continue using other elementary matrices to show $A = \lambda I$.

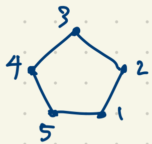
$G = GL_n(F)$ is generated by elementary matrices so $A \in Z(G)$ iff A commutes with all elementary matrices.

$Z(G)$ might be trivial e.g. $Z(S_3) = \{()\}$.

Another construction of subgroups: Suppose $G \leq S_n$. So G permutes $[n] = \{1, 2, \dots, n\}$.

The <u>stabilizer</u> of a point $x \in [n]$ is $\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\} \leq G$.

Eg.



The symmetry group of a regular pentagon is a group G which is dihedral of order 10 (sometimes denoted D_5 or D_{10}).

$$G = \{\underbrace{(), (12345), (13524), (14253), (15432)}_{5 \text{ rotations}}, \underbrace{(12)(35), (13)(45), (14)(23), (15)(24), (25)(34)}_{5 \text{ reflections}}\}$$

$G \leq S_5$ permuting $[5] = \{1, 2, 3, 4, 5\}$, the five vertices.

$\mathrm{Stab}_G(3) = \{(), (15)(24)\}$.

$()(x) = x$

If $g, h \in \mathrm{Stab}_G(x)$ then

$(gh)(x) = g(h(x)) = g(x) = x$

If $g \in \mathrm{Stab}_G(x)$ then $g(x) = x$ so

$x = g^{-1}(g(x)) = g^{-1}(x)$ so $g^{-1} \in \mathrm{Stab}_G(x)$.

Elements of order 2 in a group are called involutions.

If G is abelian then the product of any two involutions in G has order ≤ 2. If $|a| = |b| = 2$ then $(ab)^2 = abab = a^2b^2 = 1\cdot 1 = 1$ so $|ab| = 1$ or 2. If $ab = 1$ then $b = a$; otherwise $ab \ne 1$, $(ab)^2 = 1$ so ab is an involution so $\{1, a, b, ab\} = \langle a, b\rangle$ is a Klein four-subgroup of G. Any two distinct involutions in an abelian group G generate a Klein four-subgroup.

$(12)(13) = (132)$ in S_3.

How many involutions can a finite abelian group G have?

If G has k involutions then every involution lies in exactly $\frac{k-1}{2}$ Klein four-subgroup

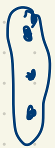
How many Klein four-subgroups does G have altogether?

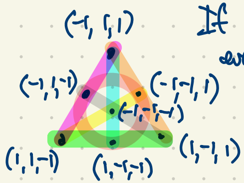
Count subgroups of the form $\langle a,b\rangle = \{1, a, b, ab\}$ where $a, b \in G$ are distinct involutions

k choices for a

$k-1$ choices for b

$\frac{k(k-1)}{6}$ is the number of Klein four-subgroups in G.





If $k = 7$ then we have 7 involutions, 7 Klein four-subgroups, every involution is in 3 Klein four-groups, every Klein four-group has 3 involutions.

In a direct product of three groups of order two e.g. $\langle -1\rangle \times \langle -1\rangle \times \langle -1\rangle$

$= \{(x, y, z) : x, y, z \in \langle -1\rangle\}$

$\langle -1\rangle = \{1, -1\}$

Certainly $k \equiv 1$ or $3 \mod 6$

In general if a,b are distinct involutions in a group G then what can they generate?

$\langle a,b\rangle = \{1, a, b, ab, ba, aba, bab, abab, baba, \dots \}$ with possible duplicates.

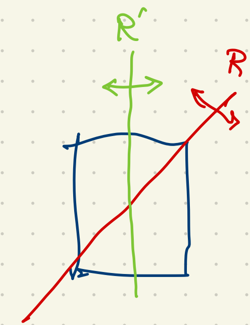
The symmetry group of an infinite string $\dots T\,T\,T\,T\,T\,T\,T \dots$ is generated by two reflections a,b in vertical axes l, l' as shown

ab is a translation (shift) one step to the right
ba is a translation one step to the left.

$\langle ab\rangle = \{\dots, baba, ba, 1, ab, abab, ababab, \dots\}$ is an infinite cyclic group, a subgroup of $\langle a,b\rangle$

$\langle a,b\rangle$ itself is an infinite dihedral group.

The symmetry group of a square is a dihedral group $\langle R,R'\rangle$ generated by two reflections



$\langle R,R'\rangle = \{ I, R, R', RR', R'R, RR'R, R'RR', RR'RR' \}$

$RR'RR' = R'RR'R$

Comments on HW2:

Recall in class we used the product π of elements in a finite abelian group.

#5(a) Show that π has order ≤ 2.

<u>Proof</u> If $G = \{g_1, g_2, \dots, g_n\}$ is abelian of order n then $\pi = g_1 g_2 \cdots g_n = g_1^{-1} g_2^{-1} \cdots g_n^{-1}$ so

$\pi^2 = (\cancel{g_1}\cancel{g_2}\cdots \cancel{g_n})(\cancel{g_1^{-1}}\cancel{g_2^{-1}}\cdots \cancel{g_n^{-1}}) = e$ (the identity element of G).

Eg. G is cyclic of order 4.

In multiplicative notation, $G = \langle g \rangle = \{1, g, g^2, g^3\}$ where $g^4 = 1$; $\pi = 1 \cdot g \cdot g^2 \cdot g^3 = g^2$ of order 2.

In additive notation, $G = \mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\} = \langle 1 \rangle$; $\pi = 0+1+2+3 = 2$ of order 2. 0 is the identity.

In S_4, $G = \langle (1234) \rangle \le S_4$, $\pi = () \circ (1234) \circ (13)(24) \circ (1432) = (13)(24)$ of order 2.

$\lim_{x\to\infty} \frac{\sin x}{x} = 0$. $\lim_{\varepsilon\to\infty} \frac{\sin\varepsilon}{\varepsilon} = 0$ is problematic in its unorthodox choice of variable $\varepsilon \to \infty$.

$\lim_{\varepsilon\to 0} \frac{\sin\varepsilon}{\varepsilon} = 1$ is natural.

$G = SL_2(\mathbb{F}_3) = \{2\times 2$ matrices of $\mathbb{F}_3$ having determinant $1\}$. $|G| = 24$.

Is $G \cong S_4$? G as only one involution whereas S_4 has 9 involutions. (An involution in any group element of order 2.)

If $G = SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ then G has only one involution, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$. $GL_2(\mathbb{R})$ has <u>many</u> involutions.

eg. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is a reflection in the y-axis.

Does $GL_2(\mathbb{R})$ have an element of order 11? Yes; in fact $SL_2(\mathbb{R})$ does:

$$\begin{bmatrix} \cos\frac{2\pi}{11} & -\sin\frac{2\pi}{11} \\ \sin\frac{2\pi}{11} & \cos\frac{2\pi}{11} \end{bmatrix} \in SL_2(\mathbb{R})$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix} \in GL_2(\mathbb{R}) \text{ has determinant } -1.$$

why is $-I$ the only involution in $SL_2(\mathbb{R})$?

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A is a reflection

(Infinitely many involutions in $GL_2(\mathbb{R})$.)

Conjugacy in groups

Two elements $g, h \in G$ are <u>conjugate</u> if $h = aga^{-1}$ for some $a \in G$. We write $h \sim g$ in this case.

This is an equivalence relation on G:

- For every $g \in G$, $g \sim g$. $(g = ege^{-1})$
- $g \sim h$ iff $h \sim g$. If $h \sim g$ then $h = aga^{-1}$ for some $a \in G$ so $g = a^{-1}ha = (a^{-1})h(a^{-1})^{-1}$
- If $h \sim g \sim w$ then $h \sim w$. If $h = aga^{-1}$ and $g = bwb^{-1}$ then $h = a(bwb^{-1})a^{-1} = (ab)w(ab)^{-1}$.

(Look up equivalence relations in the textbook, Math 2800, my videos)

Eg. in $GL_n(F)$, conjugacy is just similarity. Look in linear algebra textbook. Two matrices $A, B \in GL_n(F)$ are similar iff they represent the same linear transformation with respect to a different choice of basis.

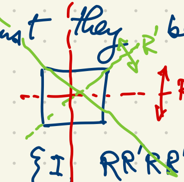
In general, conjugate elements in G have the same order. Why?

If $h = aga^{-1}$ then $h^n = \underbrace{(aga^{-1})(aga^{-1})\cdots(aga^{-1})}_{n \text{ times}} = ag^na^{-1}$. If $g^n = 1$ then $h^n = 1$. Conversely, if $h^n = 1$ then $g^n = 1$.

It follows that $|h| = |g|$ whenever h, g are conjugate in G.

Is the converse true? If two elements have the same order, must they be conjugate in G?

No; e.g. in the symmetry group of a square, $G = \langle R, R' \rangle$ where



$RR'R$ is conjugate to R'

$R'RR' = R'R(R')^{-1} \sim R$

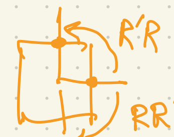
$R'RR'$ is conjugate to R

$G = \{ I, R, R', \boxed{RR', R'R,}\ RR'R, R'RR', RR'RR' \}$, $Z(G) = \langle RR'RR' \rangle = \{I, RR'RR'\}$

90° rotations

reflections

$RR'RR' = R'RR'R$ half-turn about the center



The two elements of order 4 are conjugate: $R'R = (R)RR'(R)$

The dihedral group of order 8 has conjugacy classes as follows:

$$\underset{\text{order } 1}{\{I\}},\quad \underset{\text{order } 2}{\{RR'RR'\}},\quad \underset{\text{order } 2}{\{R,\ R'RR'\}},\quad \underset{\text{order } 2}{\{R',\ RR'R\}},\quad \underset{\text{order } 4}{\{RR',\ R'R\}}.$$

If $z \in Z(G)$ then $\{z\}$ is a conjugacy class by itself $\quad (aza^{-1} = aa^{-1}z = ez = z)$

Conjugacy in S_n

eg. $\sigma = (1\ 5\ 7)(2\ 4)$, $\quad \tau = (1\ 2\ 3)(5\ 6\ 8)(4\ 7)$ in S_8

$$\begin{array}{ccccc} \sigma = (1 & 5 & 7)(2 & 4) \\ \pi\downarrow & \downarrow & \downarrow\ \ \downarrow & \downarrow \\ \pi\sigma\pi^{-1} = (6 & 4 & 2)(7 & 3) \end{array}$$
$$= (2\ 6\ 4)(3\ 7)$$

$\sigma \sim \tau\sigma\tau^{-1} = (1\ 2\ 3)(5\ 6\ 8)(4\ 7)(1\ 5\ 7)(2\ 4)(1\ 3\ 2)(5\ 8\ 6)(4\ 7) = (2\ 6\ 4)(3\ 7)$

$|\sigma| = 6$, $\quad |\tau\sigma\tau^{-1}| = 6$.

In S_n, two elements (i.e. permutations) are conjugate iff they have the same cycle structure.

In S_8, $(1\ 7\ 6\ 8\ 3\ 5)$ has order 6 but it cannot be conjugate to σ since its cycle structure is different.

Let $\pi = (1\ 6)(2\ 7)(3\ 8\ 5\ 4)$

A faster way to compute $\tau\sigma\tau^{-1}$:

$$\begin{array}{ccccc} \sigma = (1 & 5 & 7)(2 & 4) \\ \downarrow & \downarrow & \downarrow\ \ \downarrow & \downarrow\tau \\ \tau\sigma\tau^{-1} = (2 & 6 & 4)(3 & 7) \end{array}$$

$$\sigma = (1\ 5\ 7)(2\ 4)(3)(6)(8)$$
$$\rho = (2\ 6\ 4)(3\ 7)(1)(5)(8)$$

<u>Theorem</u> In S_n, two permutations are conjugate iff they have the same cycle structure (ie. the same number of cycles of each length).

<u>Proof</u> Let $\sigma, \tau \in S_n$. If $\sigma(i) = j$ then $(\tau\sigma\tau^{-1})(\tau(i)) = \tau(j)$. $\quad \left((\tau\sigma\tau^{-1})(\tau(i)) = \tau\sigma(i) = \tau(j) \right)$.

$$\sigma = (\ldots, i, j, \ldots)(\ldots)(\ldots)\ldots \iff \tau\sigma\tau^{-1} = (\ldots, \tau(i), \tau(j), \ldots)(\ldots)(\ldots)\ldots$$

Commutative Diagrams: Suppose $\sigma, \rho \in S_n$ are conjugate via $\pi \in S_n$.

$\underbrace{S_n = \mathrm{Sym}[n]}_{\text{the } n! \text{ permutations of } [n].}$, $[n] = \{1, 2, \dots, n\}$ (the n points that we are permuting)

$$\begin{tikzcd} {[n]} \arrow[r, "\sigma"] \arrow[d, "\pi"'] \arrow[dr, phantom, "//"] & {[n]} \arrow[d, "\pi"] \\ {[n]} \arrow[r, "\rho"] & {[n]} \end{tikzcd}$$

This diagram commutes: $\pi\sigma = \rho\pi$

$\iff \pi\sigma\pi^{-1} = \rho$

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] \arrow[dr] & C \arrow[r] \arrow[d] & F \\ & D \arrow[r] & E \arrow[ur] & \end{tikzcd}$$

In $GL_2(\mathbb{R})$ we have many elements of order 2.

e.g. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ $(a \in \mathbb{R})$ $\quad \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\left\{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right\}$ is a conjugacy class by itself.

If $a \in \mathbb{R}$, $A = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ then A is diagonalizable. Look for eigenvalues and eigenvectors of A.

The characteristic polynomial of A is $\det(xI - A) = \det \underbrace{\begin{bmatrix} x-1 & -a \\ 0 & x+1 \end{bmatrix}}_{\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}} = (x+1)(x-1) = x^2 - 1$. The eigenvalues of A are ± 1.

$\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector for eigenvalue 1 iff $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix}$; take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector for eigenvalue -1 iff $\begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$. Take $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{a}{2} \\ 1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -a/2 \\ 1 \end{bmatrix} = \begin{bmatrix} a/2 \\ -1 \end{bmatrix}$

$A \underbrace{\begin{bmatrix} 1 & a/2 \\ 0 & -1 \end{bmatrix}}_{M \text{ (having the eigenvectors as its columns)}} = \begin{bmatrix} 1 & -a/2 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & a/2 \\ 0 & -1 \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \Rightarrow A = MDM^{-1}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "D"] \arrow[d, "M"'] & \mathbb{R}^2 \arrow[d, "M"] \\ \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \end{tikzcd}$$

A is similar (i.e. conjugate) to D

$AM = MD \Rightarrow A = MDM^{-1}$.

A matrix in $GL_2(\mathbb{R})$ is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ iff it has trace 0 and determinant -1.