Algebra I

Group Theory

Book 2

Transpositions $(i\ j)$ are odd permutations.

$$(1\,2\,3\,4\,5\,6\,7\,8\,9) = (1\,9)(1\,8)(1\,7)(1\,6)(1\,5)(1\,4)(1\,3)(1\,2)$$

A k-cycle is a product of $k-1$ transpositions.
If k is even, this is odd; and vice versa.
A cycle of odd length is an even permutation;
〃 〃 〃 even 〃 〃 〃 odd 〃 .

If α is a product of an even numbes of transpositions, then α is an even permutation.
〃 〃 〃 〃 〃 〃 〃 odd 〃 〃 〃 〃 〃 〃 〃 〃 odd 〃 .

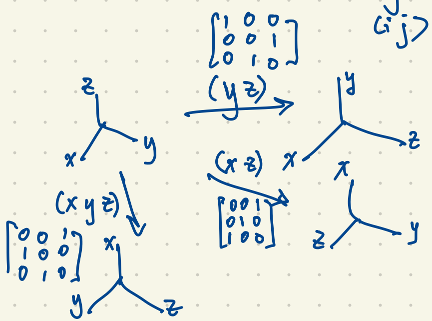
Permutations in S_5:

Even		Odd	
$()$	1	$(i\ j)$	10
$(i\ j\ k)$	20	$(i\ j\ k\ l)$	30
$(i\ j\ k\ l\ m)$	24	$(i\ j\ k)(l\ m)$	20
$(i\ j)(k\ l)$	15		
	60		60

$|S_5| = 120$

$A_5 = \{\text{even permutations in } S_5\}$

$|A_5| = 60$



An even permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-preserving transformation.
An odd permutation of the coordinate axis in $\mathbb{R}^n$ is an orientation-reversing transformation.
If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation then

$$\det T \begin{cases} = 0 & \text{if } T \text{ is not invertible} \\ > 0 & \text{〃 〃 preserves orientation} \\ < 0 & \text{〃 〃 〃 reverses 〃} \end{cases}$$

A permutation $\alpha \in S_n$ can be expressed as a product of transpositions.

If α is a product of an even number of transpositions, then α is even.

- - - - - . . odd . . . . — — — — - - . odd.

In S_3:

$(1\,3)(1\,2)(1\,3)(2\,3)(2\,3)(2\,3)(1\,2)(2\,3) = (1\,2\,3)$ says $(1\,2\,3)$ is an even permutation.

$S_3 \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right\rangle \cong$ dihedral group of order 6 (symmetry group of an equilateral triangle)

n	no. of groups of order n up to isomorphism
1	1
2	1
3	1
4	2
5	1
6	2
7	1
8	5

Groups of order 2

$S_2 \cong \{0, 1\}$ mod 2 under addition $\cong \langle -1 \rangle$ under multiplication

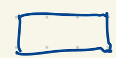
$\circ$	$()$	(12)
$()$	$()$	(12)
(12)	(12)	$()$

$+$	0	1
0	0	1
1	1	0

$\cdot$	1	-1
1	1	-1
-1	-1	1

Cayley tables of groups of order 2 all "look the same"

has a cyclic symmetry group of order 4



has an abelian symmetry group of order 4 which is not cyclic (the Klein four-group)

<u>Theorem</u> Any two groups of prime order are isomorphic; they are cyclic of order p.