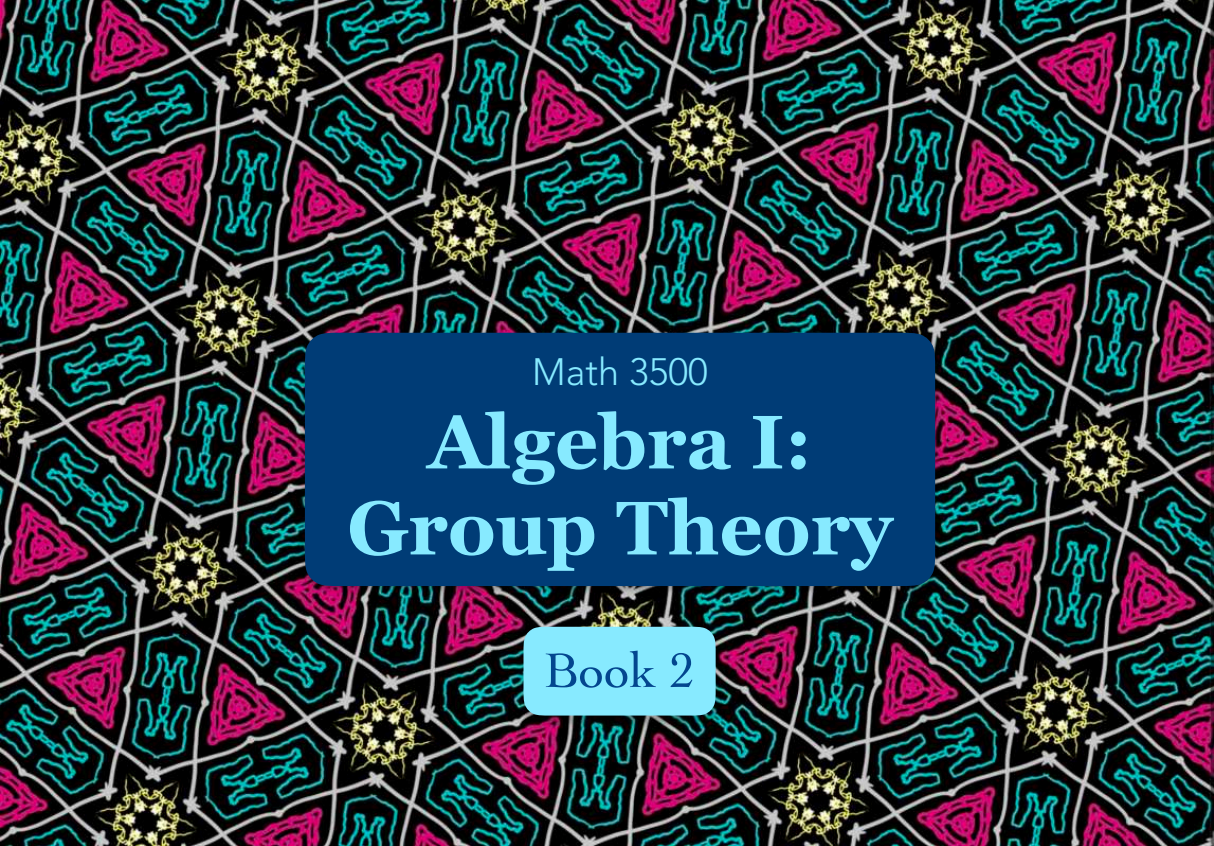




Math 3500

Algebra I: Group Theory

Book 2

Similar to HW#2: How many elements of each order does S_4 have?

1. element of order 1 : $(\quad) = \text{identity}$

1 element of order 1 : $() = \text{identity}$
 9 elements of order 2 : $(12), (13), (14), (23), (24), (34),$ $\leftarrow \binom{4}{2} = 6$ transpositions
 $\leftarrow \frac{1}{2!} \binom{4}{2} \binom{2}{2} = \frac{6}{2} = 3$

8 elements of order 3: $(12)(34), (13)(24), (14)(23), (123), (124), (132), (134), (142), (143), (234), (243)$

6 elements of order 4: $(1234), (1243), (1324), (1342), (1423), (1432)$

$$24 = 4! = |S_4|$$

$$|S_n| = n! = 1 \times 2 \times 3 \times \dots \times n$$

In S_n the number of n -cycles is $(n-1)!$

the binomial coefficient $\binom{n}{k}$ ("n choose k") is the number of ways to choose a subset of size k from a set of size n. By the way, $\binom{n}{k} =$

$\binom{n}{k}$ = k^{th} entry in n^{th} row of Pascal's Triangle

$$\binom{4}{2} = \text{number of 2-subsets in } [4] = \{1, 2, 3, 4\}$$

By the way, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Theorem $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$

Pascal's Triangle

Diagram illustrating the construction of Pascal's triangle using the addition rule. The rows are labeled $n=0$ through $n=7$. The numbers in each row are calculated by adding the two numbers directly above them. The diagonal lines are labeled $k=1$ through $k=7$.

$n=0$	1
$n=1$	1, 1
$n=2$	1, 2, 1
$n=3$	1, 3, 3, 1
$n=4$	1, 4, 6, 4, 1
$n=5$	1, 5, 10, 10, 5, 1
$n=6$	1, 6, 15, 20, 15, 6, 1
$n=7$	1, 7, 21, 35, 35, 21, 7, 1

A transposition is a 2-cycle $(i\ j) \in S_n$, $i \neq j$ in $[n] = \{1, 2, \dots, n\}$.

Products of disjoint transpositions eg. $(1\ 3)(2\ 5)(6\ 8) \in S_8$
are elements of order 2.

How many elements of order 2 are there in S_7 ?

Transpositions: $(12), (13), (14), \dots, (67)$ i.e. $(i\ j)$ where $i \neq j$ in $[7] = \{1, 2, \dots, 7\}$

$$\binom{7}{2} = 21 \text{ transpositions}$$

Products of two disjoint transpositions eg. $(26)(34) = (34)(26)$

$$\text{Number of these is } \frac{1}{2} \underbrace{\binom{7}{2}}_{21} \underbrace{\binom{5}{2}}_{10} = 105$$

Products of three disjoint transpositions eg. $(1\ 5)(2\ 7)(3\ 6) = (15)(36)(27) = (27)(36)(15)$

$$\text{Number of these is } \frac{1}{6} \underbrace{\binom{7}{2}}_{21} \underbrace{\binom{5}{2}}_{10} \underbrace{\binom{3}{2}}_3 = \frac{21 \times 10 \times 3}{6 \cdot 2} = 105$$

Number of 3-cycles in S_7 eg. (274) : $2\binom{7}{3} = 2 \times 35 = 70$

Number of products of two disjoint 3-cycles: eg. $(274)(356) = (356)(274)$

$$\frac{1}{2} \cdot 70 \cdot \binom{4}{3} \cdot 2 = 70 \cdot 4 = 280$$

Elements of order 12 in S_7 eg. $(142)(3756)$

$$70 \cdot 3! = 70 \cdot 6 = 420 \text{ elements of order 12 in } S_7$$

$$280 + 70$$

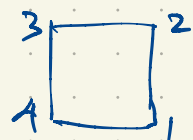
$$= 350$$

elements
of order 3
in S_7

Revisiting the dihedral group of order 8 (symmetry group of a square)

$$G = \{ I, R, R^2, R^3, D, D', V, H \}$$

viewed as a group of permutations of the four vertices



$$I \mapsto ()$$

$$R \mapsto (1234)$$

$$R^2 \mapsto (13)(24)$$

$$R^3 \mapsto (1432)$$

$$H \mapsto (12)(34)$$

$$V \mapsto (14)(23)$$

$$D \mapsto (13)$$

$$D' \mapsto (24)$$

$$G \cong \text{subgroup of } S_4 : \{ (1), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24) \}$$

$$RV = (1234)(14)(23) = (24) = D' = \langle (1234), (13) \rangle$$

This is an example of a permutation group of degree 4, i.e. a subgroup of S_4 .
A permutation group of degree n is a subgroup of the symmetric group S_n .

Theorem (Cayley's Representation Theorem) Every finite group is isomorphic to a permutation group. In fact, if $|G| = n$ then G is isomorphic to a subgroup of S_n . (But we can usually do better. Eg. the dihedral group of order 8 is isomorphic to a subgroup of S_8 . But S_4 is even better.)

The two most important general classes of examples of groups are
(i) permutation groups (i.e. subgroups of S_n) and
(ii) linear groups (i.e. subgroups of $GL_n(F) = \{ \text{invertible } n \times n \text{ matrices over a field } F \}$). eg. $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$ (integers mod p)

The dihedral group of order 8 is also a subgroup of $GL_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ at } bc \neq 0 \right\}$

$I: \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$R: \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$R^2: \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$R^3: \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$H: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$G = \text{symmetry group of square} \cong \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}$

$D: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$D': \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

R^2 commutes with every element of G (not so immediately obvious from the other ways of representing G).

(the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ rotated 90° counter-clockwise about the origin)

subgroup of $GL_2(\mathbb{R})$

$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle$

Similar to HW2 #4, 5: $\mathbb{F}_p = \{0, 1, \dots, p-1\}$ integers mod p .

eg. $p=3$, $F = \mathbb{F}_3 = \{0, 1, 2\}$

$$\begin{array}{c|ccc} + & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array}$$

$$\begin{array}{c|ccc} \times & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{array}$$

$$\frac{1}{2} = 2 \quad -\frac{1}{2} = -2 = 1$$

$$F^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in F \right\}$$

$$|F^2| = 3^2 = 9$$

2×2 matrices over F there are $3^4 = 81$ 2×2 matrices over F .

$$GL_2(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in F, \quad \underline{ad - bc \neq 0} \right\}$$

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

ie. $ad - bc = 1 \text{ or } 2$ ie. $ad - bc = \pm 1$.

$$|GL_2(F)| = 8 \times 6 = 48$$

$9-1=8$ choices for first column to be nonzero
 $9-3=6$ choices for the second column to be not a scalar multiple of the first column.

eg. $g = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \in GL_2(F) \Rightarrow g^2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$
 so $|g| = 2$.

Eg. $F = \mathbb{F}_2 = \{0, 1\}$ $\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$ $\begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$ ie. $\det A = 1$
 Let's try $G = GL_3(F) = \{3 \times 3 \text{ matrices } A \text{ over } F \text{ such that } \det A \neq 0\}$

The total number of 3×3 matrices over F is $2^9 = 512$.

$$|G| = 7 \times 6 \times 4 = 168$$

$\swarrow$ $\nwarrow$ $\nwarrow$
 $8-1$ nonzero vectors $8-2$ $8-4$

$$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

F^3 is a 3-dimensional vector space over F
 $|F^3| = 2^3 = 8$

eg. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in G$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}_{A^2} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^4 = A^2 A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$|A| = 4 = \text{the order of } A$
 Note: $4 \mid 168$